MONOCHROMATIC LINES IN PARTITIONS OF Z^N

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INTRODUCTION

It is well known that perfect play in the familiar game of Tic-Tac-Toe* always results in a draw. It is less well known (but equally true) that no draw is possible for the 3-dimensional analogue of Tic-Tac-Toe. More precisely, if the set of integer points $I_3^3 = \{(x_1,x_2,x_3):x_i \varepsilon \{0,1,2\}, 1 \leq i \leq 3\}$ is arbitrarily partitioned into two classes, say $I_3^3 = C_1 \cup C_2$, then at least one of the classes must contain three collinear points. However, it is easy to construct partitions of $I_4^3 = \{(x_1,x_2,x_3):x_i \varepsilon \{0,1,2,3\}$ into two classes, neither of which contains four collinear points.**

In this paper we would like to describe recent work directed towards sharpening the known bounds for the n-dimensional generalizations of this problem. In particular, we show that in n dimensions, there exists a board of size O(n) for which no draw is possible.

Because of space requirements, and keeping more in line with the talk on which this paper is based, we will furnish detailed proofs for very few of the assertions made. Rather we will indicate the underlying ideas needed and the general techniques used. Full details for these assertions (as well as their generalizations to the rather more difficult case of partitions into q classes for an arbitrary prime power q) can be found in [11].

* Also known as Noughts and Crosses in some parts of the world.

** The 4 × 4 × 4 analogue of Tic-Tac-Toe, marketed under the name of Qubic, has recently been shown by O. Patashnik to be a win for the first player. For an interesting account of this difficult computation, the reader should consult [16].

PRELIMINARIES

We first formulate the problem we study more precisely. For an arbitrary (fixed) integer $n \geq 2$, let $\mathbb{Z}^n$ denote the set of integer points in Euclidean n-space. A geometric line L of length ℓ in $\mathbb{Z}^n$ is defined to be a set of points described by

$$L = \{(x_1,\ldots,x_n): x_i = c_i + d_i u, u = 1,2,\ldots,\ell\}$$

where

(1) $$\text{g.c.d.}\{d_1,d_2,\ldots,d_n\} = 1.$$

The special lines in Tic-Tac-Toe have all $d_i = 0$ or ± 1. Condition (1) just guarantees that any lattice point in the convex hull of L is also in L.

By a 2-coloring χ of $\mathbb{Z}^n$, we just mean a map $\chi:\mathbb{Z}^n \to \{0,1\}$. A subset $X \subseteq \mathbb{Z}^n$ is said to monochromatic* under χ if for some $i \in \{0,1\}$,

$$X \subseteq \chi^{-1}\{i\}.$$

For a 2-coloring χ of $\mathbb{Z}^n$, let $\ell(\chi)$ denote the length of the longest monochromatic line in $\mathbb{Z}^n$. Finally, define

(2) $$\rho(n) = \inf_{\chi} \ell(\chi)$$

where χ ranges over all 2-colorings of $\mathbb{Z}^n$.

It follows from the fundamental result of Hales and Jewett [12], [10] that

(3) $$\rho(n) \to \infty \text{ as } n \to \infty.$$

Essentially, this theorem asserts the following: For any integers r and t, there is an integer $N = N(r,t)$ so that in any r-coloring of $I_t^N = \{(x_1,\ldots,x_N): x_i = 0,1,\ldots,t-1, 1 \leq i \leq N\}$ there is always a monochromatic line of length t with all $d_i = 0$ or 1.

The best bounds currently known for $N(r,t)$, as well as related corollaries such as van der Waerden's theorem for arithmetic progressions, are extremely weak. We will discuss these more fully at the end of the paper.

* Also often called homogeneous.

Our goal will be to bound $\rho(n)$ from above.

THE LINEAR UPPER BOUND

It turns out the basic functions we will use in our proofs depend on very old and fundamental quantities in combinatorics, namely, the binomial coefficients. However, we will derive several (what we believe to be) new results concerning them which are of interest* in themselves.

Let $\mathbb{Z}_2 = \{0,1\}$ denote the field of two elements.

Definition: For $a \geq 0$, define $g_a\colon\ \mathbb{Z} \to \mathbb{Z}_2$ by

(4) $$g_a(x) \equiv \binom{x}{a} \equiv \frac{x(x-1)\dots(x-a+1)}{a!} \pmod 2$$

In Table 1 we list some of the initial values of the g_a.

a \ x	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	0	1	0	1	0	1	0	1
2	0	0	1	1	0	0	1	1
3	0	0	0	1	0	0	0	1
4	0	0	0	0	1	1	1	1
5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	0	1	1
7	0	0	0	0	0	0	0	1

$g_a(x)$

Table 1

We next list various facts concerning the g_a. Let us write

$$a = \sum_{i\geq 0} a_i 2^i,\ x = \sum_{i\geq 0} x_i 2^i,\ \text{etc.},$$

in their binary expansions.

Fact 1. $g_a(x) = 1$ if and only if $x_i \geq a_i$ for all i.

* In fact, perhaps of more interest than the main results of the paper.

Proof: Since the exact power of 2 which divides m! is $\sum_{k\geq 1} \left[\frac{m}{2^k}\right]$ then $\binom{a+b}{b}$ is odd if and only if

(5) $$\sum_{k\geq 1} \left[\frac{a+b}{2^k}\right] = \sum_{k\geq 1} \left[\frac{a}{2^k}\right] + \sum_{k\geq 1} \left[\frac{b}{2^k}\right].$$

But

$$[\alpha+\beta] \geq [\alpha] + [\beta]$$

implies that (5) holds if and only if

$$\left[\frac{a+b}{2^k}\right] = \left[\frac{a}{2^k}\right] + \left[\frac{b}{2^k}\right] \text{ for all } k.$$

Thus, (5) holds iff there is no carrying when adding a and b written base 2. Therefore, $g_a(x) = 1$ iff $x_i \geq a_i$ for all i. ∎

From Fact 1, a number of very useful results follow.

Fact 2. If $2^t \leq a < 2^{t+1}$ then g_a has period 2^{t+1}, i.e., $g(x+2^{t+1}) = g(x)$ for all $x \in \mathbf{Z}$.

Fact 3.

$$g_a(x) = \begin{cases} 0 \text{ for } x = 0,1,\ldots,a-1, \\ 1 \text{ for } x = a. \end{cases}$$

It follows from Facts 2 and 3 that the g_a, $0 \leq a < 2^{t+1}$, are independent over $\mathbf{Z}_2$ and, in fact, form a basis for functions $f\colon \mathbb{Z} \to \mathbb{Z}_2$ which have period 2^{t+1}.

It is clear that $g_a(x+1)$ has the same period as $g_a(x)$. More precise information is given in the following.

Fact 4.

$$g_a(x+1) \equiv g_a(x) + \sum_{i<a} \varepsilon_i g_i(x) \pmod 2$$

for a suitable choice of $\varepsilon_i = \varepsilon_i(a) \in \mathbf{Z}_2$.

Similarly, the period of $g_a(dx)$ divides the period of $g_a(x)$. It is not too difficult to prove the following crucial result.

Fact 5.

$$g_a(dx) \equiv \begin{cases} g_a(x) + \sum\limits_{i<a} \varepsilon_i(d) g_i(x), & d \text{ odd} \\ \sum\limits_{i \le a/2} \varepsilon_i(d) g_i(x), & d \text{ even} \end{cases}$$

for suitable $\varepsilon_i(d) \in \mathbb{Z}_2$.

Finally, we have the very useful product formula.

Fact 6 (orthogonality). If $a + b < 2^{t+1}$ then

$$\sum_{x=0}^{2^{t+1}-1} g_a(x) g_b(x) \equiv \begin{cases} 1 \text{ if } a + b = 2^{t+1} - 1, \\ 0 \text{ otherwise.} \end{cases}$$

The preceding facts can now be used to prove the following result.

Lemma. If $\varepsilon_a \not\equiv 0 \pmod 2$ then

$$f(x) \equiv \sum_{i \le a} \varepsilon_i g_i(x) \pmod 2$$

can have at most a consecutive equal values.

Proof: Assume the contrary and suppose without loss of generality (by Fact 4) that

(6) $$f(2^{t+1}-a-1) = \ldots = f(2^{t+1}-2) = f(2^{t+1}-1) = 0.$$

By Fact 3,

$$g_b(0) = g_b(1) = \ldots = g_b(b-1) = 0.$$

Thus, for $2^{t+1} - a - 1 = b$,

$$0 = \sum_{x=0}^{2^{t+1}-1} f(x) g_b(x) = \sum_{x=0}^{2^{t+1}-1} \sum_{i=0}^{a} \varepsilon_i g_i(x) g_b(x)$$

$$= \sum_{i=0}^{a} \varepsilon_i \sum_{x=0}^{2^{t+1}-1} g_i(x) g_b(x)$$

$$= \varepsilon_a$$

by Fact 6, which contradicts the initial hypothesis on ε_a. ∎

We are now in a position to prove a linear upper bound on $\rho(n)$. This result first appeared in [18]. Define a 2-coloring χ of the points $\bar{x} = (x_1,\ldots,x_n) \in \mathbf{Z}^n$ by

(6) $$\chi(\bar{x}) = \sum_{k=1}^{n} g_{n-1+k}(x_k) \pmod 2$$

Then

(7) $$\ell(\chi) \leq 2n-1.$$

To see this, consider a line

$$L: x_i(u) = c_i + d_i u, \quad u = 0,1,2,\ldots$$

where g.c.d.$\{d_1,d_2,\ldots,d_n\} = 1$. Thus, some d_i is odd. Let k be the largest index with d_k odd. Then

$$\chi(\bar{x}(u)) = \sum_{i=1}^{n} g_{n-1+i}(x_i(u)) = \sum_{i=1}^{n} g_{n-1+i}(c_i+d_i u)$$

$$= g_{n-1+k} + \sum_{i<n-1-k} \varepsilon_i g_i(u)$$

for suitable $\varepsilon_i \in \mathbf{Z}_2$ since by Fact 5, the $g_{n-1+i}(c_i+d_i u)$ with d_i even collapse into $\mathbb{Z}_2$-linear combinations of $g_j(u)$'s with $j \leq \frac{1}{2}(n-1+i) < n \leq n-1+k$. Thus, by the Lemma, $\chi(\bar{x}(u))$ has at most

$n - 1 + k \leq 2n - 1$ consecutive equal values. This proves (7). It follows from (7) that

$$(8) \qquad \rho(n) \leq 2n - 1$$

which is the upper bound promised for this section. Note that (7) actually holds when g.c.d.$\{d_1,\ldots,d_n\}$ is odd.

A SUBLINEAR UPPER BOUND

The basic idea we will use in reducing the bound in (8) is to replace the terms $g_j(x)$ in the definition of χ by functions $h_j(x,y,\ldots,z)$ of many variables which behave in certain ways like $g_j(x)$. The key result for such a substitution is the following extension of Fact 6.

Fact 7 (generalized orthogonality). If $0 \leq a,b,\ldots,c < 2^{t+1}$ then

$$\sum_{x=0}^{2^{t+1}-1} g_a(x)g_b(x)\ldots g_c(x) \equiv \begin{cases} 0 \text{ if } a_i+b_i+\ldots+c_i = 0 \text{ for some } i, \\ 1 \text{ if } a_i+b_i+\ldots+c_i = 1 \text{ for all } i. \end{cases}$$

As an example of the type of substitution we have in mind, consider the function h_5 defined by

$$h_5(x_1,x_2) \equiv g_5(x_1) + g_5(x_2) + g_1(x_1)g_4(x_2) (\text{mod } 2).$$

Suppose $\bar{x}(u) = (x_1(u),\ldots,x_N(u))$ with

$$x_i(u) = c_i + d_iu, \text{ g.c.d.}\{d_1,\ldots,d_N\} = 1$$

and

$$\chi(\bar{x}) = \ldots + h_5(x_1,x_2) + \ldots .$$

Then

$$\chi(\bar{x}(u)) = \ldots + g_5(c_1+d_1u) + g_5(c_2+d_2u)$$
$$+ g_1(c_1+d_1u)g_4(c_2+d_2u) + \ldots .$$

Therefore,

$$\sum_{u=0}^{2^{t+1}-1} \chi(\bar{x}(u))g_2(u)$$

$$= \ldots + \sum_{u=0}^{2^{t+1}-1} h_5(c_1+d_1u, c_2+d_2u)g_2(u) + \ldots$$

$$= \ldots + \sum_{u=0}^{2^{t+1}-1} g_5(c_1+d_1u)g_2(u) + \sum_{u=0}^{2^{t+1}-1} g_5(c_2+d_2u)g_2(u)$$

$$+ \sum_{u=0}^{2^{t+1}-1} g_1(c_1+d_1u)g_4(c_2+d_2u)g_2(u) + \ldots .$$

By Fact 7, the sum of the three displayed terms is 1 modulo 2 unless $d_1 \equiv d_2 \equiv 0 \pmod 2$. Thus, if the line $\bar{x}(u)$ moves non-trivially in either the x_1 or the x_2 direction then $g_5(u)$ occurs in the expansion of $\chi(\bar{x}(u))$ with a coefficient of 1. It is in this sense that the use of h_5 in the definition of the coloring χ is equivalent to the use of g_5. However, h_5 "uses up" two coordinates where as g_5 only uses up one. For $k = \sum_{i\geq 0} k_i 2^i$, let $w(k)$ (called the weight of k) be defined by $w(k) = \sum_{i\geq 0} k_i$. Define $h_k : \mathbf{Z}^{w(k)} \to \mathbb{Z}_2$ by

$$h_k(z_1,\ldots,z_{w(k)}) = \sum_I G_I$$

where I ranges over all nonempty subsets of $\{1,\ldots,w(k)\}$ and $G_I = \prod_{i\in I} g_{a_i}(z_i)$ with $\sum_{i\in I} a_i = k$ having no carries when performed to the base 2. It is not hard to show in this case (using most of the preceding Facts) that if

$$z_i(u) = c_i + d_i u \text{ and } 2^t \leq k < 2^{t+1}$$

then

$$\sum_{u=0}^{2^{t+1}-1} h_k(z_1(u),\ldots,z_{w(k)}(u))g_{2^{t+1}-1-k}(u) \equiv 0 \pmod 2)$$

if and only if all the d_i are <u>even</u>.

The coloring χ^* we use for the sublinear bound can now be described. Let $W(n)$ denote $\sum_{j=1}^{n} w(j)$ and set $N(n) = W(2n-1) - W(n-1)$. Define

$$\chi^*: \mathbb{Z}^{N(n)} \to \mathbb{Z}_2 \text{ by}$$

$$\chi^*(z_1,\ldots,z_{N(n)}) = \sum_{k=n}^{2n-1} h_k(z_{W(k-1)-W(n-1)+1},\ldots,z_{W(k)-W(n-1)})$$

It follows from the preceding Facts and the definition of h_k (in much the same way as in the proof of (7)) that if g.c.d.$\{d_1,\ldots,d_{N(n)}\} = 1$ then in the expansion of $\chi^*(\bar{z}(u))$ as a $\mathbb{Z}_2$-linear combination of g_k's, some g_i with $i \leq 2n - 1$ occurs non-trivially. The Lemma then implies that $\chi^*(\bar{z}(u))$ has at most $2n - 1$ equal values. Thus,

$$\ell(\chi^*) \leq 2n - 1$$

and consequently,

(9) $$\rho(N(n)) \leq 2n - 1.$$

Finally, since it can be shown that (cf. [3])

$$W(m) = (1+o(1)) \frac{m \log m}{2 \log 2}$$

then straightforward computation using (9) implies the main result of the paper:

<u>Theorem</u>.

(10) $$\rho(n) \leq (1+o(1)) \frac{2n \log 2}{\log n}$$

CONCLUDING REMARKS

As we remarked earlier, a similar but more complicated analysis can be carried out (see [11]) with an arbitrary prime power q replacing 2. These results imply the following upper bound on $\rho_r(n)$, the quantity which corresponds to $\rho(n)$ when r colors are used in coloring $\mathbb{Z}^n$.

$$\rho_r(n) \leq (1+o(1)) \frac{2 \log r}{r-1} \cdot \frac{n}{\log n}.$$

It is interesting to note that the properties of the g_a expressed in Facts 2, 3 and 4 actually characterize $\binom{x}{a}$ (mod 2). In fact, for an arbitrary prime p, suppose $f_a : \mathbf{Z} \to \mathbb{Z}_p$, $a = 0,1,2,\ldots$, is a sequence of functions satisfying:

(i) $f_a(x)$ has period p^{t+1} where $p^t \leq a < p^{t+1}$; $f_0(x)$ has period 1;

(ii) $f_a(x) = \begin{cases} 0 \text{ for } x = 0,1,\ldots,a-1, \\ 1 \text{ for } x = a, \end{cases}$

(iii) $f_a(x+1)$ is a $\mathbf{Z}_p$-linear combination of $f_i(x)$, $0 \leq i \leq a$.

Then (see [11])

$$f_a(x) \equiv \binom{x}{a} \pmod p.$$

This is not the case when p is composite.

A fundamental question in this subject which at present remains completely unanswered is whether or not the <u>density</u> version of the Hales-Jewett theorem holds. To explain what we mean by this, consider the well known theorem of van der Waerden on arithmetic progressions (see [22], [9], [10]):

> For all integers k and r, there is a number $W(k,r)$ so that in any r-coloring of $\{1,2,\ldots,W(k,r)\}$ there is always a monochromatic arithmetic progression of k terms.

Observe that this result follows from the Hales-Jewett theorem - simply associate the integers in $[0,k^n-1]$ with $\mathbb{Z}_k^n$ by $x = \sum_{i=1}^{n} x_i k^{i-1} \leftrightarrow (x_1,x_2,\ldots,x_n)$. Nearly 50 years ago, Erdös and Turán [4] raised the question of determining <u>which</u> color class contains the long arithmetic progressions. In particular, they conjectured that the "most frequently occurring" color should have this property. This was shown to be the case for 3-term progressions by Roth [19] and finally, in 1974, Szemerédi managed to prove the general result.

<u>Theorem</u>* (Szemerédi [21]). For all k and $\varepsilon > 0$ there is a number

* Recently, Furstenburg and others (see [5], [6]) have succeeded in proving Szemerédi's theorem using newly developed results from ergodic theory.

$S(k,\varepsilon)$ such that if $R \subseteq \{1,2,\ldots,S(k,\varepsilon)\}$ and $|R| \geq \varepsilon\{S(k,\varepsilon)\}$ then R must contain a k-term arithmetic progression.

Van der Waerden's theorem is an example of a Ramsey (or partition) theorem. Szemerédi's theorem (which clearly implies van der Waerden's result) is the stronger <u>density</u> version of it.

It is very tempting to believe that the corresponding density version of the Hales-Jewett theorem should hold.

<u>Conjecture</u>*. For all t and $\varepsilon > 0$, there exists a number $C(t,\varepsilon)$ so that if $n \geq C(t,\varepsilon)$ and $R \subseteq \{(x_1,\ldots,x_n): x\varepsilon\{0,1,\ldots,t-1\}\}$ has $|R| \geq \varepsilon t^n$ then R contains a line of length t of the form $x_i = c_i + d_i u$ with all $d_i = 0$ or 1.

Even the case $t = 2$, the only non-trivial case for which the conjecture is known to be true, requires an argument. In this case, we are required to find a "line" which consists of two points of the form:

$$\begin{cases} x = (\ldots,a,\ldots,0,\ldots,b,\ldots,0,\ldots,c,\ldots), \\ \\ y = (\ldots,a,\ldots,1,\ldots,b,\ldots,1,\ldots,c,\ldots). \end{cases}$$

However we can associate to each point $z = (z_1,z_2,\ldots,z_n)$, $z_i = 0$ or 1, a subset $Z = \{1,2,\ldots,n\}$ in the usual way; namely, $i \varepsilon Z$ iff $z_i = 1$. Under this association, our "line" is just a pair of subsets X, Y with X a proper subset of Y. But a theorem of Sperner [20] shows that the largest family of subsets of $\{1,2,\ldots,n\}$ having no member properly contained in another has at most $\binom{n}{\left[\frac{n}{2}\right]} \sim \frac{2^n}{\sqrt{\pi n}} = o(2^n)$ members. Thus, if $|R| > \varepsilon 2^n$ then it must contain a line of the desired form.

Very recently, progress has been made by T. C. Brown and J. P. Buhler (see [2]) for a weakened version of the case $t = 3$.

Finally, we make a few remarks concerning what we believe to be the "truth" concerning the actual values of $\rho(n)$. From above it seems likely that $\rho(n) = o(\log n)$ or perhaps even $\rho(n) = o(\log\log n)$ should be true. However, the lower bound for $\rho(n)$, which depends on the known lower bounds for the Hales-Jewett theorem, is embarrassingly weak (it is not even primitive recursive). As an example of a measure of our ignorance in this area, consider the following related Ramsey-type theorem.

* One of the authors is currently offering US $1000 for a resolution of this conjecture. It is actually a strengthened version of an earlier conjecture of Moser [15].

<u>Theorem</u> (see [8]). Let $Q_n = \{(x_1,\ldots,x_n): x_i = 0 \text{ or } 1\}$. Then there is a number N_0 so that for any 2-coloring of the line segments joining pairs of points in Q_n, there always exist four <u>coplanar</u> points of Q_n spanning 6 line segments all having the same color.

The best estimate from above currently available for N_0 can be described as follows (also see [1], [7], [14]).

Following Knuth [13], define

$$a\uparrow n = a^n,$$

$$a\uparrow\uparrow n = \underbrace{a\uparrow(a\uparrow(\ldots(a\uparrow a))\ldots)}_{n\ a\text{'s}},$$

$$= a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}\Big\}\, n$$

and, in general,

$$a\overbrace{\uparrow\uparrow\ldots\uparrow}^{t+1} n = a\overbrace{\uparrow\ldots\uparrow}^{t}(a\overbrace{\uparrow\ldots\uparrow}^{t}(\ldots(a\overbrace{\uparrow\ldots\uparrow}^{t} a)\ldots)).$$

where n a's occur on the right-hand side. (The reader is invited to something as simple as $3\uparrow\uparrow\uparrow\uparrow 3$ into normal notation). Then it has been shown that

$$N_0 \le \left.\begin{array}{c} 3\uparrow\uparrow\uparrow\uparrow 3 \\ \overbrace{3\uparrow\uparrow\ldots\uparrow 3} \\ \overbrace{3\uparrow\uparrow\uparrow\quad\uparrow\uparrow 3} \\ \cdots\cdots \\ \overbrace{3\uparrow\uparrow\uparrow\uparrow\ldots\ldots\uparrow\uparrow\uparrow\uparrow 3} \end{array}\right\}\ \text{64 layers}$$

where each number represents the number of <u>arrows</u> in the expression below it.

The best lower bound known for N_0 is:

$$N_0 \ge 6.$$

Probably, $N_0 = 6$.

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