Proc. Natl. Acad. Sci. USA
Vol. 78, No. 7, p. 4001, July 1981
Mathematics

On irregularities of distribution of real sequences

(approximation/clustering)

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Communicated by David Slepian, April 13, 1981

ABSTRACT **A natural measure of the amount of unavoidable clustering that must occur in any bounded infinite sequence of real numbers is studied. We determine the extreme value for this measure and exhibit sequences that achieve this value.**

A fundamental problem in the study of the distribution of sequences is to obtain precise estimates for the extent that any sequence must deviate from some appropriately defined standard of regularity. Among the many results available in this topic (see ref. 1), those of Roth (2) and Schmidt (3) are particularly noteworthy.

In this announcement, we consider the following question: How much "clustering" must occur in an arbitrary real sequence $\bar{x} = (x_0, x_1, \ldots)$ with $x_i \in [0,1]$, in which the clustering of $\bar{x}$ is measured by

$$C(\bar{x}) = \inf_n \liminf_{m\to\infty} n\,|x_{m+n} - x_m|.$$

The rationale behind this natural definition of irregularity [suggested by a question of D. J. Newman (see ref. 4)] is clear. If $\bar{x}$ were somehow perfectly spread out, one might hope that $|x_{m+n} - x_m| \geq 1/n$ for all m and n (and indeed, there are $\bar{x}$ for which this happens for all m and infinitely many n). A related but less sensitive measure was previously investigated by de Bruijn and Erdös in ref. 5. They showed that, for any sequence $\bar{x}$ in $[0,1]$,

$$\hat{C}(\bar{x}) \equiv \liminf_{n\to\infty} \min_{0\leq i<j\leq n} n|x_i - x_j| \leq 1/\log 4,$$

and further, that the constant $1/\log 4$ is best possible. Observe that for all $\bar{x}$, $C(\bar{x}) \leq \hat{C}(\bar{x})$.

Our first result furnishes a precise bound for $C(\bar{x})$.

THEOREM 1. *For any sequence* $\bar{x}$ *in* $[0,1]$,

$$C(\bar{x}) \leq \left(1 + \sum_{k\geq 1} F_{2k}^{-1}\right)^{-1} \equiv \alpha = 0.39441967\ldots, \qquad [1]$$

in which F_n *denotes the* nth *Fibonacci number, defined by* $F_0 = 0$, $F_1 = 1$, *and* $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$.

The bound 1 is best possible, as shown by the next result. For each integer $n \geq 0$, let $\varepsilon(n)$ denote the unique sequence $(\varepsilon_1(n), \varepsilon_2(n), \ldots) = (\varepsilon_1, \varepsilon_2, \ldots)$ satisfying:

(*i*) $n = \sum_{i\geq 1} \varepsilon_i F_{2i}$;

(*ii*) For all i, $\varepsilon_i = 0$, 1, or 2;

(*iii*) If $\varepsilon_i = \varepsilon_j = 2$, $i < j$, then for some k with $i < k < j$, $\varepsilon_k = 0$.

Define the sequence $\bar{x}^* = (x_0^*, x_1^*, \ldots)$ by

$$x_n^* = \alpha \sum_{i\geq 1} \varepsilon_i(n) F_{2i}^{-1}.$$

Note that $x_n^* \in [0,1]$ and that $\bar{x}^*$ is nowhere dense.

THEOREM 2.

$$C(\bar{x}^*) = \alpha \qquad [2]$$

In fact,

$$\inf_n \inf_m n|x_{m+n}^* - x_m^*| = \alpha. \qquad [3]$$

Theorems 1 and *2* are intimately tied to the following extremal result on the set S_m of permutations on $\{1,2,\ldots,m\}$. Define

$$u_m \equiv \min_{\pi\in S_m} \max_I \sum_k |\pi(i_{k+1}) - \pi(i_k)|^{-1},$$

in which I ranges over all increasing subsequences $\{i_1 < i_2 < \ldots < i_r\} \subseteq \{1,2,\ldots,m\}$.

THEOREM 3.

$$u_m = \begin{cases} 1 + \sum_{k=1}^{t} F_{2k}^{-1} & \text{if } F_{2t+3} \leq m < F_{2t+4}, \\ 1 + \sum_{k=1}^{t} F_{2k}^{-1} + F_{2t+3}^{-1} & \text{if } F_{2t+4} \leq m < F_{2t+5}. \end{cases} \qquad [4]$$

Permutations for which equality in 3 holds can be generated by the ordering of the first m terms of $\bar{x}^*$. These are the same permutations formed by the well-known sequence $\{k\tau\}$, $k = 0,1,2,\ldots$, in which $\tau = (1 + \sqrt{5})/2$ and $\{x\}$ denotes the fractional part of x.

The proofs of the preceding results are somewhat delicate and rather lengthy and will be given elsewhere.

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