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ON STEINER TREES FOR BOUNDED POINT SETS

1. Introduction

There are many situations in which one would like to connect a set of points X together with a network having a total length as short as possible. Such networks, called *minimum Steiner trees* for X, have a long and venerable history, dating back to Steinhaus, Maxwell and, in a primitive form, even to Fermat (see [3]). For precise definitions and a brief summary of recent work on the subject the reader may consult [4].

In this note we study the following question, first raised by L. Few [2] in 1955: What is the greatest[1] length $s(n)$ a minimum Steiner tree for a set of n points contained in a unit square can have?

By considering subsets of a regular hexagonal lattice placed in a unit square, it is easy to show that

(1) $$s(n) \geqslant (\tfrac{3}{4})^{1/4} n^{1/2} + O(1).$$

In [2], Few succeeded in proving

(2) $$s(n) \leqslant n^{1/2} + \tfrac{7}{4}.$$

Since $(\frac{3}{4})^{1/4} = 0.9306\ldots$, there was still considerable room for improvement in the coefficient of $n^{1/2}$. However, no progress was made on this problem for some 25 years.

The present authors' attention was drawn to this question by L. Mirsky [5] who suggested that perhaps it was time to try to correct this unsatisfactory state of affairs. This note represents a first step in this direction. In particular we prove

(3) $$s(n) < 0.995 n^{1/2}$$

for n is sufficiently large.

We also consider the corresponding question when the metric in question is not the Euclidean (or L_2) metric but rather the rectilinear (or L_1) metric given by

$$d^*((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

This is often the most appropriate distance measure for designing printed circuit boards (and traveling in large cities). For this version. we show that the corresponding maximum length $s^*(n)$ satisfies

(4) $$s^*(n) \leqslant n^{1/2} + 1 + o(1).$$

[1] Standard compactness arguments show that $s(n)$ is well defined.

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Since examples formed from subsets of a square lattice show that

(5) $$s^*(n) \geqslant n^{1/2} + O(1)$$

then the bounds on $s^*(n)$ are quite tight. In fact, when $n = r^2$, we are able to determine $s^*(n)$ exactly. In this case

(6) $$s^*(r^2) = r + 1.$$

We conjecture that

(7) $$s^*(n) \leqslant n^{1/2} + 1$$

for all n.

2. Few's Proof

We begin by reviewing the elegant argument Few uses in [2] to prove (2). For an integer m (to be specified later), place $m + 1$ equally-spaced horizontal lines L_i on the unit square S (see Figure 1).

The spacing between adjacent lines is $1/m$. Also, we denote the vertical boundary lines V_0 and V_1 as shown in Figure 1. Consider an arbitrary fixed set of n points p_k placed in S. For each point p_k, join it with vertical segments to the nearest two lines L_i and L_{i+1}. The total length of all these vertical line segments, all the L_i and V_0 and V_1 is just

$$(m + 1) + 2 + n(1/m).$$

However, note that $V_0, L_0, L_2, L_4, \ldots$ and the vertical segments from all the p_k going to the L_{2k} form a connected network (or *Steiner tree*) joining all the p_k. Similarly, so do $V_1, L_1, L_3, L_5, \ldots$ and the vertical segments from all the p_k to the L_{2k+i}. Since these two Steiner trees are disjoint and the sum of their lengths is $m + n/m + 3$ then one of them has length at most

(8) $$\tfrac{1}{2}(m + n/m + 3).$$

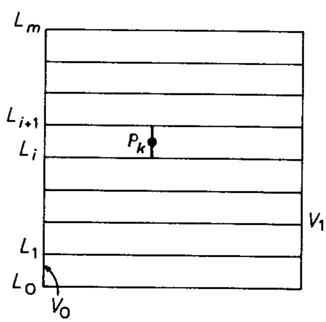

Fig. 1.

By choosing $m = [n^{1/2}]$ we obtain

(9) $$s(n) \leqslant n^{1/2} + O(1).$$

More careful analysis of the diophantine constraints yields (2).

3. A SHORTER STEINER TREE

In order to improve the coefficient of $n^{1/2}$ in (9), we shall use the following strategy. S will be subdivided into small rectangles. The two Steiner trees previously constructed will be modified depending on the local configurations of the p_k within these rectangles. As a result of the modifications, each p_k will be assigned a *savings*, which will represent a lower bound on the decrease in length of the modified Steiner tree which connects p_k and its neighbors. The sum of the savings will then be sufficient to improve the upper bound of $s(n)$.

The details of the construction are as follows. Consider a portion of S as shown in Figure 2. The height of the rectangle R is $2\alpha = 2/m$; the width of R is $\beta = 1/100\,\mathrm{m}$. The (closed) shaded portions of R will be termed the *outside* parts of R. The unshaded portions of R will be termed the *inside* parts of R. Each $L_i, 2 \leqslant i \leqslant m-1$, will bisect exactly $1/\beta = 100\,\mathrm{m}$ such adjacent rectangles (see Figure 3).

Thus, almost all points of S belong to *two* rectangles R, although almost all points of S are in an *outside part* of a unique rectangle R.

We now assume that an arbitrary fixed set P of n points $p_k, 1 \leqslant k \leqslant n$, have been placed in S. Let us examine the local situation within one of the rectangles R. There are several possibilities:

(i) Some outside part of R contains a point q_1 of P but no part of R contains more than one point of P (see Figure 4).

The line segments from the $q_j, j \neq 1$, to L_i divide the portion of L_i in R into at most four pieces. Remove the largest piece and connect q_1 to its two former endpoints by a minimum length network for these three points.

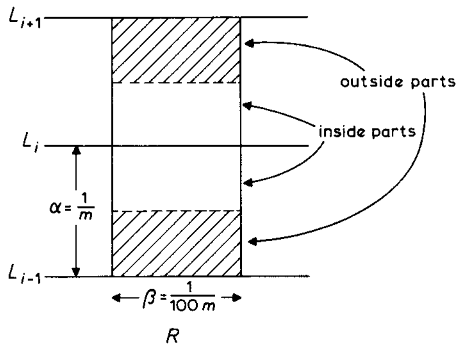

Fig. 2.

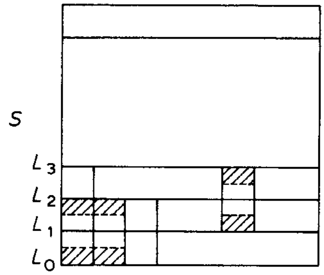

Fig. 3.

It is well known [3] that this will consist of three concurrent line segments each meeting the other two at 120°.

(ii) Some outside part of R contains more than one point, say $q_1, \ldots, q_s$, of P (see Figure 5).

In this case, connect the outside points q_j to the horizontal line segment D separating the outside and inside part of R, and join D to L_i.

It is now a simple matter to estimate the length saved in each of these cases. It turns out that in (i) we save at least

$$s_i = \frac{\alpha}{2} + \frac{\beta}{4} - \left(\left(\frac{7}{8}\beta\right)^2 + \left(\frac{\sqrt{3}}{2}\cdot\frac{\beta}{4} + \frac{\alpha}{2}\right)^2\right)^{1/2} \tag{10}$$

units of length. We shall assign $s_1/2$ of this amount to the point q_1 and the other $s_1/2$ of this amount to the point q_4, the other outside point in R if it exists.

Similarly, in (ii), it is easy to see that if there are $s \geqslant 2$ points $q_1, \ldots, q_s$ in the same outside part of R, then the construction indicated saves a total length of at least

$$s_2 = \frac{\alpha}{2}(s-1) - \beta. \tag{11}$$

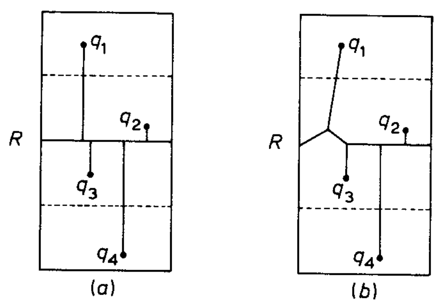

Fig. 4.

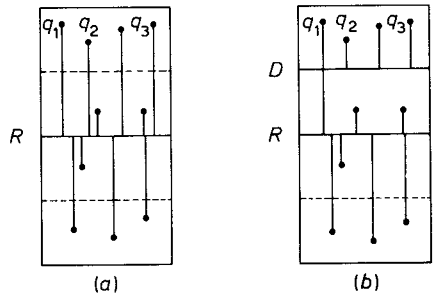

Fig. 5.

This amount of savings will be partitioned into $s + 2$ equal parts; each of the s points $q_1, \ldots, q_s$ gets assigned one part as does each of the (possibly two) single points in outside regions of rectangles in which $q_1, \ldots, q_s$ are inside points.

Now, since $\beta = \alpha/100$, the savings per point in case (i) is at least $(0.0001294)\alpha$; in case (ii) it is at least $(0.1)\alpha$ which is much larger than $(0.0001294)\alpha$.

It is important to note here that in the preceding procedure, *every* point p_k gets assigned a savings of at least $(0.0001294)\alpha = (0.0001294)/m$ since it is always an outside point of *some* rectangle or it is very near the boundary of S and can be treated as an outside point. Thus, the total length of the two Steiner trees is at most

$$(12) \qquad m + 3 + \frac{n}{m}(1 - .0001294).$$

It is easy to verify that the expression in (12) can be made less than $1.9999n^{1/2}$ for a suitable m with n sufficiently large.

Thus, at least *one* of two Steiner trees has length less than $0.99995n^{1/2}$ for n large, i.e.,

$$(13) \qquad s(n) < 0.99995n^{1/2}.$$

By a considerably more careful analysis of this general type (in which the rectangles are now thrown away but the concepts of inside and outside regions are retained) it is possible to prove

$$(14) \qquad s(n) < 0.995n^{1/2}$$

for n sufficiently large. However, the details are rather complicated and we omit them.

4. Rectilinear Steiner Trees

Consider now a set of n points $p_1, \ldots, p_n$ lying in the $a \times b$ rectangular region $R(a, b)$ shown in Figure 6 where without loss of generality, we assume $a \leqslant b$.

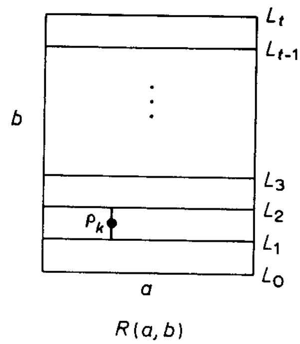

Fig. 6.

We subdivide $R(a, b)$ into t rectangular regions R_i by $t-1$ equally spaced horizontal line segments L_i. For each p_k, say $p_k \in R_i$, let V_k denote the vertical line segment through p_k bounded by L_{i-1} and L_i. Observe that the union of all the $L_i, 0 \leqslant i \leqslant t$, all the $V_k, 1 \leqslant k \leqslant n$, and the two vertical sides of $R(a, b)$ form two disjoint rectilinear Steiner trees for the p_k. The sum of their lengths is

$$(15) \qquad (t+1)a + n\frac{b}{t} + 2b.$$

Our next step is to modify the two trees we have just constructed. Suppose there is some L_i which is not connected to any V_k. We will then delete L_i. The resulting configuration can be seen to be the union of two rectilinear Steiner trees for the p_k having total length at most

$$(16) \qquad (t+1)a + n\frac{b}{t} + b.$$

Suppose now that every L_i is connected to some V_k. Note that if some L_i has no p_k on it and is connected to some V_k it may be moved vertically, say to L_i' (adjusting all the V_k accordingly to V_k') without increasing the total tree length until, in its new position, there is now at least one point p_i on it. (This applies to L_0 and L_t as well.) Let us make such transformations recursively until we reach a configuration of lines $L_i', 0 \leqslant i \leqslant t$, with each L_i' containing one of the original points p_i', and so that the total length of the two Steiner trees thus formed is still bounded above by (15). By construction, each p_i' is connected by its V_i' to either L_{i-1}' or L_{i+1}'. Let us extend each $V_i', i \neq 0, t$, to form V_i'' so that it contains p_i' and extends from L_{i-1}' to L_{i+1}'. The total increase in length due to the replacement of all the V_i' by the V_i'' does not exceed b. Finally, remove the two vertical sides of $R(a, b)$, a total length of $2b$. The resulting configuration is again easily seen to be the union of two rectilinear Steiner trees for the p_k, now having a total length not exceeding the values in (16). Thus, *some* rectilinear Steiner tree for the p_k has a length at most of

$$\frac{1}{2}\left(a+b+at+\frac{bn}{t}\right). \tag{17}$$

Note that if we choose $n=(ax+1)(bx+1)$ for integers a, b, x and place these n points in $R(a,b)$ in a regular square lattice packing with spacing $1/x$ then the length of the minimum rectilinear Steiner tree is clearly $abx + a + b$. However, by (17) we have as upper bound in this case

$$\min_t \tfrac{1}{2}\left(a+b+at+\frac{b(ax+1)(bx+1)}{t}\right)=abx+a+b$$

by choosing $t=bx+1$. Thus, (17) gives the *exact* answer for all n of the form $(ax+1)(bx+1)$.

More generally, by taking

$$t=\left(\frac{bn}{a}\right)^{1/2}+o(1)$$

we see that (17) is bounded above by

$$(abn)^{1/2}+\tfrac{1}{2}(a+b)+o(1).$$

For $a=b=1$ this implies

$$s^*(n) \leqslant n^{1/2}+1+o(1) \tag{18}$$

as claimed earlier.

TABLE I

n	$1+\frac{1}{2}\min_t\left(t+\frac{n}{t}\right)$	$s^*(n)$
2	5/2	2
3	11/4	2
4	3	3
5	13/4	4
6	7/2	10/3
7	11/3	7/2
8	23/6	11/3
9	4	4
10	17/4	4

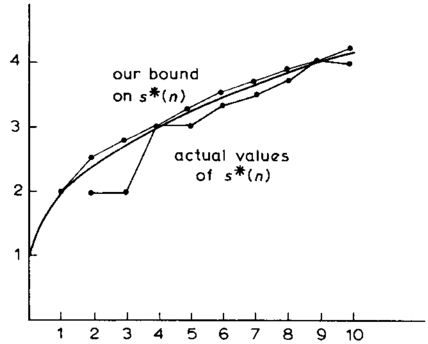

Fig. 7.

In general, for $a = b = 1$ and

$$n = r^2 + m, \quad -r < m \leqslant r,$$

by choosing $t = r$ in (17) we obtain

$$s^*(r^2 + m) \leqslant r + 1 + \frac{m}{2r}, \quad -r < m \leqslant r, \tag{19}$$

which gives a more precise version of (18).

In Table I, we list the known values of $s^*(n)$ (taken from [1]) and the upper bounds we obtain from (17). In Figure 7, we compare our bounds with the known values of $s^*(n)$ and with the values of $n^{1/2} + 1$. The bounds we get from (17) all lie on tangents to the parabola $y = x^{1/2} + 1$ with points of tangency $(r^2, r + 1)$

5. Concluding remarks

All known evidence points to the validity of the conjecture

$$s^*(n) \leqslant n^{1/2} + 1, \quad n \geqslant 2,$$

but at present we do not see how to prove this. It also appears that $s^*(r^2) = s^*(r^2 + 1) = r + 1$ for all r. However, it can be shown that $s^*(r^2 + 2) > r + 1$. Whether these are the only cases of equality is not known.

It would be nice to improve the bound on $s(n)$ to at least $s(n) < 0.99n^{1/2}$ but it seems that new ideas will be needed.

Finally, we remark that the corresponding questions can be asked for regions that are not squares, and also other metric spaces as well, but we do not consider these question here.

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