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On the Permanents of Complements of the Direct Sum of Identity Matrices

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INTRODUCTION

A classic experiment used in testing for ESP abilities has the following general form. A deck of cards, consisting of a_i identical cards of type i, $1 \leq i \leq r$, is shuffled and placed face down (in most such experiments $r = 5$ and $a_i = 5$ for $1 \leq i \leq 5$). The subject then attempts to correctly guess the type of each card as the cards are sequentially removed from the deck. In previous work [2, 3], several of the authors have analyzed the effects of allowing various kinds of feedback into this process. For example, after each incorrect guess the subject might be told what the guessed card actually was. Obviously such information, if used appropriately, could significantly increase the number of correct guesses the subject could expect to make during a pass through the deck. Consider the standard deck consisting of 25 cards, 5 cards of each of 5 types. Without any feedback (or ESP ability) the expected number of correct guesses is 5. With complete feedback, a subject can expect to achieve more than 8.64 correct guesses, simply by always guessing the most frequently occurring type in the remaining deck (see [3]).

Another very important type of feedback, investigated in [3], was that in which the subject was just told whether each guess is right or wrong (but *not* the correct identity of an incorrectly guessed card). The optimal strategy for using this kind of partial feedback is extremely complex and, in some cases, counter-intuitive. For example, the optimal strategy can require guessing a type which is *not* the most likely type in the remaining deck (see [3]).

A fundamental quantity in these studies is $N(a_1, \ldots, a_r; b_1, \ldots, b_r)$ which is defined to be the number of arrangements of a deck of $a_1 + \cdots + a_r = n$ cards, with a_i of type i, such that symbol 1 does not appear in the first b_1

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positions, symbol 2 does not appear in positions $b_1 + 1, \ldots, b_2$, etc. Usually we abbreviate the vectors $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_r)$ by $\bar{a}$ and $\bar{b}$, respectively. Then $N(\bar{a}; \bar{b})$ is simply the number of permutations of the deck for which it is possible for the subject to have (the first) b_1 guesses of type 1 be incorrect, (the next) b_2 guesses of type 2 be incorrect, etc. (A moment's reflection shows that the *order* in which the guesses are made is irrelevant as far as evaluating $N(\bar{a}; \bar{b})$ is concerned.) When $b_1 + \cdots + b_r = n$, $N(\bar{a}; \bar{b})$ divided by $n!$ equals the probability of "no matches" in the following card matching experiment: Let deck 1 contain a_i cards of type i, deck 2 contain b_i cards of type i. Both decks are shuffled and cards are turned up in pairs, simultaneously.

It is this combinatorial quantity $N(\bar{a}; \bar{b})$ this paper will investigate. Specifically we will derive various explicit expressions along with numerous monotonicity and unimodality properties for the $N(\bar{a}; \bar{b})$. These results have been used in the evaluation of feedback experiments (see [2, 3]).

It turns out that the $N(\bar{a}; \bar{b})$ actually occur in a variety of guises throughout combinatorics, e.g., in the study of rook polynomials, permutations with restricted positions, enumeration of systems of distinct representatives, and the evaluation of (0,1)-permanents. Thus, our results for $N(\bar{a}; \bar{b})$ have applications to these areas as well.

The function of $N(\bar{a}; \bar{b})$ was first discussed by Kaplansky [8, 9] who describes some applications. Kaplansky's work has been extended recently by Even and Gillis [5] along with Askey and Ismail [1]. These authors provide an interesting representation of $N(\bar{a}; \bar{b})$ in terms of Laguerre polynomials and a list of related references. The main new results of this paper are the inequalities, but several of the preliminary results, as well as the applications are also novel.

Other Interpretations of $N(\bar{a}; \bar{b})$

Let us form an n by n matrix $M(\bar{a}; \bar{b})$ as shown in Fig. 1. Thus, $M(\bar{a}; \bar{b})$ consists of all 1's except for disjoint blocks of 0's of sizes a_i and b_i. Suppose we now identify the first a_1 rows of $M(\bar{a}; \bar{b})$ with the a_1 cards of type 1, the next a_2 rows of $M(\bar{a}; \bar{b})$ with the a_2 cards of type 2, etc. This gives a natural ordering of the deck. Let us identify the columns of $M(\bar{a}; \bar{b})$ with the n guesses we will (eventually) make. Then each arrangement of the deck corresponds to permutation choice from $M(\bar{a}; \bar{b}) = (m_{ij})$, i.e., a choice of n entries no two being in the same row or column, as follows: If the sth card of the deck is in the tth position of the arrangement then the entry m_{st} belongs to the permutation choice. In order for an arrangement to be consistent with having the first b_1 guesses being incorrect type 1 guesses, the next b_2 guesses being incorrect type 2 guesses, etc., it is necessary and sufficient that the corresponding permutation choice contains none of the

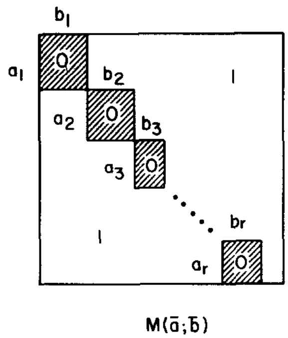

FIGURE 1

0's in the a_i by b_i 0-subblocks of $M(\bar{a};\bar{b})$. Thus, a permutation choice $m_{i,\pi(i)}$, $1 \le i \le n$, from $M(\bar{a};\bar{b})$ corresponds to a consistent arrangement of the deck iff

$$\prod_{i=1}^{n} m_{i,\pi(i)} \neq 0.$$

Since each $m_{ij} = 0$ or 1, the *number* of consistent arrangements is just

$$\sum_{\pi} \prod_{i=1}^{n} m_{i,\pi(i)}, \tag{1}$$

where π ranges over all permutations of $\{1,2,\ldots,n\}$. The expression in (1) is exactly the definition of the permanent of $M(\bar{a};\bar{b})$. Therefore we have:

Fact 1.

$$N(\bar{a};\bar{b}) = \operatorname{Per} M(\bar{a};\bar{b}).$$

We point out that this could actually be taken as the definition of $N(\bar{a};\bar{b})$. It is useful when $N(\bar{a};\ \bar{b})$ is undefined, e.g., when $a_1 + \cdots + a_r < b_1 + \cdots + b_r$.

If we picture $M(\bar{a};\bar{b})$ as a generalized n by n chessboard in which the cells corresponding to 0's are forbidden then by Fact 1, $N(\bar{a};\bar{b})$ is just the number of ways of placing n nonattacking rooks on this restricted board $B = B(\bar{a};\bar{b})$. Hence, if we let $R_B(x)$ denote the ordinary *rook polynomial* $R_B(X) = \sum_{i=0}^{n} p_i x^i$ associated with B (where p_i is the number of ways of placing i nonattacking rooks on the board) then

$$N(\bar{a};\bar{b}) = p_n.$$

(For a detailed exposition of the many interesting properties of rook polynomials, the reader should consult [6] or [13]).

Finally, let $S_1,\dots,S_m$ be a family $\mathbb{S}$ of subsets of $\{1,2,\dots,n\}$. A *system of distinct representatives* (SDR) of $\mathbb{S}$ is a 1-1 mapping $\lambda:\{1,2,\dots,m\}\to\{1,2,\dots,n\}$ such that

$$\lambda(i)\in S_i \qquad \text{for all } i\in\{1,2,\dots,m\}.$$

Take $m=n$ and define

$$S_i=\{1,2,\dots,b_1+\cdots+b_k\}\cup\{b_1+\cdots+b_k+b_{k+1}+1,\dots,b_1+\cdots+b_r\}$$

for $a_1+\cdots+a_k+1\le i\le a_1+\cdots+a_k+a_{k+1}$. This defines a family $\mathbb{S}=\mathbb{S}(\bar{a};\bar{b})$ of n sets S_i which has the following property: Each SDR of $\mathbb{S}$ corresponds to a unique permutation choice from $M(\bar{a};\bar{b})$. Therefore, we have

Fact 2. $N(\bar{a};\bar{b})$ is exactly the number of SDR's of the family $\mathbb{S}(\bar{a};\bar{b})$. (For further information on SDR's the reader is referred to [1, 12, 13]).

Elementary Properties of $N(\bar{a};\bar{b})$

If $b_1=\cdots=b_r=0$ then $M(\bar{a};\bar{0})$ is the all 1's matrix and by Fact 1,

$$N(\bar{a};\bar{0})=n!$$

Of course, in general, $N(\bar{a};\bar{b})\le n!$ since the permanent of a nonnegative matrix cannot increase when positive entries are replaced by 0.

On the other hand, if for some i, $a_i+b_i>n$ then it follows, that $N(\bar{a};\bar{b})=0$. This can be seen by observing that there are just $n-a_i$ cards which are not of type i. Thus, if $b_i>n-a_i$ guesses of type i are made, at least $b_i-(n-a_i)>0$ must be correct. In particular, they cannot all have been incorrect. Hence, there are *no* consistent arrangements of the deck, i.e., $N(\bar{a};\bar{b})=0$.

The next result shows that the converse holds.

Theorem 1. $N(\bar{a};\bar{b})=0$ *iff for some* i,

$$a_i+b_i>n. \tag{2}$$

Proof. We have already seen the "if" direction. To prove the "only if" direction, suppose

$$a_i+b_i\le n \qquad \textit{for } 1\le i\le r.$$

We show that $N(\bar{a};\bar{b})>0$. To do this, we use the SDR interpretation. By Fact 2, it is enough to show that the family $\mathbb{S}(\bar{a};\bar{b})$ has at least one SDR. By the Hall "Marriage Theorem" (see [12]) this is equivalent to showing that for any k distinct sets $S_{i_1},\dots,S_{i_k}$ of $\mathcal{S}(\bar{a};\bar{b})$, $1 \le k \le n$,

$$\left|\bigcup_{j=1}^{k} S_{i_j}\right| \ge k. \tag{3}$$

There are two possibilities. If not all the S_{i_j} are equal, say, $S_{i_1} \ne S_{i_2}$, then from the definition of the S_i,

$$S_{i_1} \cup S_{i_2} = \{1,2,\dots,n\}$$

and

$$\left|\bigcup_{j=1}^{k} S_{i_j}\right| = n \ge k$$

as required. On the other hand, suppose all the S_{i_j} are equal, say, to

$$S = \{1,2,\dots,n\} - \{b_1 + \cdots + b_m + 1,\dots,b_1 + \cdots + b_m + b_{m+1}\}.$$

Thus,

$$|S| = n - b_{m+1} \qquad \text{and} \qquad k \le a_{m+1}.$$

Consequently, by (2)

$$\left|\bigcup_{j=1}^{k} S_{i_j}\right| = n - b_{m+1} \ge a_{m+1} \ge k$$

and the theorem is proved. □

The Basic Recurrence

For an arbitrary fixed $k \in \{1,2,\dots,n\}$, let us denote by $\bar{\delta}_k$ the vector $(0,0,\dots,1,\dots,0)$ which has a 1 in the kth component and 0's everywhere else. Suppose at some point during the guessing experiment, the subject has made only incorrect guesses, which we represent as usual by the vector $\bar{b} = (b_1,\dots,b_r)$. There are two possibilities for the next card:

(i) It is not of type k. Thus, if the next guess were type k, it would be incorrect. Consequently the number of arrangements for which this can happen is $N(\bar{a};\bar{b} + \bar{\delta}_k)$.

(ii) It is of type k. In this case, since there are a_k cards of type k currently in the deck then the number of arrangements for which this can happen is $a_k N(\bar{a} - \bar{\delta}_k; \bar{b})$.

Since these two cases are exhaustive, we can write the following recurrence for $N(\bar{a};\bar{b})$.

$$N(\bar{a};\bar{b}) = N(\bar{a};\bar{b} + \bar{\delta}_k) + a_k N(\bar{a} - \bar{\delta}_k;\bar{b}) \qquad (4)$$

for any $k \in \{1,2,\ldots,n\}$ for $b_1 + \cdots + b_r < n$. Of course, this can also be easily seen from the permanent interpretation of $N(\bar{a};\bar{b})$. Although (4) can be used to derive a closed form expression for $N(\bar{a};\bar{b})$ (we do this in the next section), it seems to be more efficient to use (4) directly for computing large sets of values of $N(\bar{a};\bar{b})$. We list some small values in the Appendix.

Explicit Forms for $N(\bar{a};\bar{b})$

A formula for $N(\bar{a};\bar{b})$ was first derived by Kaplansky [8, 9]. See also Askey and Ismail [1]. We give a different derivation of this formula. To solve the recurrence (4) subject to the initial condition $N(\bar{a};\bar{0}) = n!$, define

$$Q(\bar{a};\bar{b}) = N(\bar{a};\bar{b})/a_1!\ldots a_r!.$$

Thus, Q satisfies

$$Q(\bar{a};\bar{b} + \bar{\delta}_k) = Q(\bar{a};\bar{b}) - Q(\bar{a} - \bar{\delta}_k;\bar{b}), \qquad 1 \le k \le r, \qquad (5)$$

and $Q(\bar{a};\bar{0}) = n!/a_1! \cdots a_r!$. Iterating (5) we obtain, for any integer j,

$$Q(\bar{a};j\bar{\delta}_k) = Q(\bar{a};\bar{0}) - \sum_{i=0}^{j-1} Q(\bar{a} - \bar{\delta}_k;i\bar{\delta}_k), \qquad 1 \le k \le r. \qquad (6)$$

We will consider one variable at a time in (6). To simplify notation, consider a function $Y(a;b)$ of two integer variables a and b which satisfies the following analog of (6):

$$Y(a;b) = Y(a;0) - \sum_{i=0}^{b-1} Y(a-1;i) \qquad \text{for all } a,b \ge 0, \qquad (6')$$

with $Y(x,y) = 0$ for $x < 0$. Elementary arguments show that (6′) has the solution

$$Y(a;b) = \sum_{i=0}^{a} (-1)^i \binom{b}{i} Y(a-i;0). \qquad (7)$$

Using (7) in (5) for $k = 1,2,\ldots,r$ yields

$$Q(\bar{a};\bar{b}) = \sum_{\bar{0}\le\bar{i}\le\bar{a}} (-1)^{|\bar{i}|}\binom{b_1}{i_1}\cdots\binom{b_r}{i_r}Q(\bar{a}-\bar{i};\bar{0})$$
$$= \sum_{\bar{0}\le\bar{i}\le\bar{a}} (-1)^{|\bar{i}|}\binom{b_1}{i_1}\cdots\binom{b_r}{i_r}\frac{((a_1-i_1)+\cdots+(a_r-i_r))!}{(a_1-i_1)!\ldots(a_r-i_r)!}, \tag{8}$$

where $\bar{x}\le\bar{y}$ means $x_k \le y_k$ for all k and $|\bar{i}|$ denotes $i_1 + i_2 + \cdots + i_r$. Substituting for $N(\bar{a};\bar{b})$ we obtain:

THEOREM 2 (KAPLANSKY).

$$N(\bar{a};\bar{b}) = \sum_{\bar{i}}(-1)^{|\bar{i}|}(n-|\bar{i}|)!\prod_{k=1}^{r}\binom{a_k}{i_k}\binom{b_k}{i_k}i_k! \tag{9}$$

There are several interesting observations which follow at once from the form of $N(\bar{a};\bar{b})$ in (9).

(i) When $|\bar{b}| = n$, N is symmetric in $\bar{a}$ and $\bar{b}$, i.e., $N(\bar{a};\bar{b}) = N(\bar{b};\bar{a})$

(ii) $N(\bar{a};\bar{b})$ is divisible by both $a_1!\ldots a_r!$ and $b_1!\ldots b_r!$. Since the (multiple) sum in (9) involves $\prod_{i=1}^{r}(1+\min(a_i,b_i))$ terms, it is usually not particularly convenient for calculating specific values of N.

(iii) In the special case that $\bar{a} = \bar{1}$, so that $r = n$, (9) implies

$$N(\bar{1};\bar{b}) = \sum_{k=0}^{n}(-1)^k \mathbb{S}_k(\bar{b})(n-k)!, \tag{10}$$

where $\mathbb{S}_k(\bar{b})$ is the kth elementary symmetric function of the variables $b_1,b_2,\ldots,b_n$, i.e.,

$$\mathbb{S}_1(\bar{b}) = \sum_i b_i, \mathbb{S}_2(\bar{b}) = \sum_{i<j} b_ib_j, \text{etc.}$$

This will be useful in showing (in the next section) the Schur convexity of $N(\bar{1};\bar{b})$.

In the special case that $\bar{b} = \bar{1}$ then (10) reduces to the well-known expression

$$N(\bar{1};\bar{1}) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!},$$

which counts the number of "derangements" of n elements, i.e., the number of fixed-point-free permutations on n elements (see, e.g., [11]).

(iv) Using Ryser's expansion of the permanent (see [11]) it is not difficult to derive the following expression for $N(\bar{a};\bar{b})$. Define $a_{r+1} = 0$, $b_{r+1} = n - (b_1 + \cdots + b_r)$. Then

$$N(\bar{a};\bar{b}) = \sum_{\bar{i}} (-1)^{n-|\bar{i}|} \prod_{k=1}^{r+1} \binom{b_k}{i_k} (|\bar{i}| - i_k)^{a_k}. \tag{10'}$$

In this expression the symmetry between $\bar{a}$ and $\bar{b}$ when $|\bar{b}| = n$ is not as apparent as in (9).

Other more complicated expressions for $N(\bar{a};\bar{b})$ can be derived from other expansions of the permanent, e.g., the Binet–Minc expansion (see [11]). It is an interesting combinatorial exercise to show the equality of the various expressions directly.

Inequalities for $N(\bar{a};\bar{b})$

We begin our discussion of inequalities with a motivating example due to Efron [4].

Let two decks of n cards be prepared. The first deck labeled $(1,2,\ldots,n)$, the second deck labeled $(a_1,a_2,\ldots,a_n)$ with $a_i \in \{1,2,\ldots,n\}$. Each deck is mixed and the cards turned over simultaneously, one pair at a time. Efron showed that the probability of *no* matches is largest if there are no repeated symbols among the a_i. That is, if $\{a_1,a_2,\ldots,a_n\} = \{1,2,\ldots,n\}$. Efron applied this to a problem in optimal searching. In [3] it is applied several times to prove the optimality of guessing strategies. The probability of no matches is $1/n!$ times $N(\bar{1};\bar{b})$, with b_j the number of a_i' equal to j. Thus $N(\bar{1};\bar{b})$ is largest when all b_j equal 1. This suggests that $N(\bar{1};\bar{b})$ might be Schur convex in $\bar{b}$ (see Marshall and Olkin [10, Chap. 5, Sec. D] for definitions related to Schur convexity).

Theorem 3. *$N(\bar{1};\bar{b})$ is Schur convex in $\bar{b}$.*

Proof. Assume $b_1 \geq b_2 + 2$ and let

$$\bar{b}! = (b_1',b_2',\ldots,b_n') = (b_1 - 1,b_2 + 1,b_3,\ldots,b_n)$$

and

$$\bar{b}^* = (b_3,b_4,\ldots,b_n).$$

Then by (10)

$$N(\bar{1};\bar{b}') - N(\bar{1};\bar{b}) = \sum_{k=0}^{n} (-1)^k \big(\mathbb{S}_k(\bar{b}') - \mathbb{S}_k(\bar{b})\big)(n-k)!$$

$$= \sum_{k=2}^{n} (-1)^k \big\{ b_1' b_2' \mathbb{S}_{k-2}(\bar{b}^*) - (b_1' + b_2')\mathbb{S}_{k-1}(\bar{b}^*) + \mathbb{S}_k(\bar{b}^*)$$

$$- b_1 b_2 \mathbb{S}_{k-2}(\bar{b}^*) + (b_1 + b_2)\mathbb{S}_{k-1}(\bar{b}^*) - \mathbb{S}_k(\bar{b}^*)\big\}(n-k)!$$

$$= (b_1' b_2' - b_1 b_2) \sum_{k=2}^{n} (-1)^k \mathbb{S}_{k-2}(\bar{b}^*)(n-k)! \text{ since } b_1' + b_2' = b_1 + b_2,$$

$$= (b_1' b_2' - b_1 b_2) N(\bar{1};\bar{b}^*) \geq 0 \tag{11}$$

since for $x + y$ fixed, xy increases as x and y get closer together. Since the above argument applies to any pair of coordinates b_i and b_j, $N(\bar{1};\bar{b})$ is Schur convex. Note that for $n \geq 4$, the inequality in (11) is strict. □

The next inequality for $N(\bar{a};\bar{b})$ we derive is based on the following "intuitively clear" observation: If the deck has at least as many type 2 cards as type 1 cards (i.e., $a_2 \geq a_1$) and there have been at least as many preceding incorrect guesses of type 2 as of type 1 (i.e., $b_2 \geq b_1$), then it is as least as likely that the next card is of type 2 as of type 1.

This, and more, is implied by the next result.

THEOREM 4. *For* $a_1 \leq a_2, b_1 \leq b_2$,

$$N(\bar{a} + \bar{\delta}_1;\bar{b}) - N(\bar{a} + \bar{\delta}_2;\bar{b}) \geq (b_2 - b_1) N(\bar{a};\bar{b} - \bar{\delta}_2),$$

$$N(\bar{a};\bar{b} + \bar{\delta}_1) - N(\bar{a};\bar{b} + \bar{\delta}_2) \geq (a_2 - a_1) N(\bar{a} - \bar{\delta}_2;\bar{b}). \tag{12}$$

Proof. In Fig. 2 we show the matrices $M(\bar{a} + \bar{\delta}_1;\bar{b})$ and $M(\bar{a} + \bar{\delta}_2;\bar{b})$. The two matrices differ only in that the first $b_1 + b_2$ entries in the

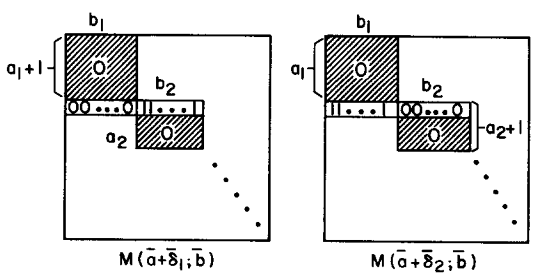

FIGURE 2

$(a_1 + 1)$st rows are complementary. Therefore,

$$\text{Per } M(\bar{a} + \bar{\delta}_1; \bar{b}) = b_2 \text{ Per } M(\bar{a}; \bar{b} - \bar{\delta}_2) + X$$

$$\text{Per } M(\bar{a} + \bar{\delta}_2; \bar{b}) = b_1 \text{ Per } M(\bar{a}; \bar{b} - \bar{\delta}_1) + X, \tag{13}$$

where we have expanded the permanents along the $(a_1 + 1)$st rows of the matrices. The quantity X depends only on portions of the matrices which are identical and is consequently the same in both expansions. Using Fact 1 we can write the fundamental relation

$$N(\bar{a} + \bar{\delta}_1; \bar{b}) - N(\bar{a} + \bar{\delta}_2; \bar{b}) = b_2 N(\bar{a}; \bar{b} - \bar{\delta}_2) - b_1 N(\bar{a}; \bar{b} - \bar{\delta}_1) \tag{14}$$

If $b_1 = 0$ then (14) reduces to:

$$N(\bar{a} + \bar{\delta}_1; \bar{b}) \geq N(\bar{a} + \bar{\delta}_2; \bar{b}) \qquad \text{for } b_1 = 0. \tag{15}$$

By symmetry we also have

$$N(\bar{a}; \bar{b} + \bar{\delta}_1) \geq N(\bar{a}; \bar{b} + \bar{\delta}_2) \qquad \text{for } a_1 = 0. \tag{15'}$$

We rewrite (14) as follows

$$N(\bar{a} + \bar{\delta}_1; \bar{b}) - N(\bar{a} + \bar{\delta}_2; \bar{b}) = (b_2 - b_1) N(\bar{a}; \bar{b} - \bar{\delta}_2) + b_1 \{ N(\bar{a}; \bar{b} - \bar{\delta}_2) - N(\bar{a}; \bar{b} - \bar{\delta}_1) \}. \tag{16}$$

Since the last term in (16) can be written

$$N(\bar{a}; \bar{b} - \bar{\delta}_2) - N(\bar{a}; \bar{b} - \bar{\delta}_1) = N(\bar{a}; \bar{b}' + \bar{\delta}_1) - N(\bar{a}; \bar{b}' + \bar{\delta}_2),$$

where $\bar{b}' = \bar{b} - \bar{\delta}_1 - \bar{\delta}_2$, and $b_1 \leq b_2$ implies $b_1' = b_1 - 1 \leq b_2 - 1 = b_2'$ then it follows by induction on $b_1 + \cdots + b_r$ that

$$N(\bar{a} + \bar{\delta}_1; \bar{b}) \geq N(\bar{a} + \bar{\delta}_2; \bar{b})$$

and

$$N(\bar{a}; \bar{b} + \bar{\delta}_1) \geq N(\bar{a}; \bar{b} + \bar{\delta}_2) \tag{17}$$

for $a_1 \leq a_2$, $b_1 \leq b_2$ (where (15) and (15′) are used to start the induction).

Applying (17) in (16) now yields

$$N(\bar{a}+\bar{\delta}_1;\bar{b}) - N(\bar{a}+\bar{\delta}_2;\bar{b}) \geq (b_2-b_1)N(\bar{a};\bar{b}-\bar{\delta}_2),$$
$$N(\bar{a};\bar{b}+\bar{\delta}_1) - N(\bar{a};\bar{b}+\bar{\delta}_2) \geq (a_2-a_1)N(\bar{a}-\bar{\delta}_2;\bar{b}) \qquad (18)$$

for $a_1 \leq a_2$, $b_1 \leq b_2$. This proves the theorem. □

Note that the "observation" mentioned before the proof follows from the second inequality of (17).

The same techniques can be used to prove the following result (where the permanents are now expanded in terms of k by k minors).

THEOREM 5. *Assume $k \leq a_1 \leq a_2$, $k \leq b_1 \leq b_2$. Then*

$$N(\bar{a}+k\bar{\delta}_1;\,\bar{b}) - N(\bar{a}+k\bar{\delta}_2;\,\bar{b}) \geq b_2^{(\underline{k})}N(\bar{a};\,\bar{b}-k\bar{\delta}_2)$$
$$- b_1^{(\underline{k})}N(\bar{a};\,\bar{b}-k\bar{\delta}_2), \qquad (19)$$

where $x^{(\underline{k})} \equiv x(x-1)\dots(x-k+1)$.

In particular

$$N(\bar{a}+k\bar{\delta}_1;\bar{b}) \geq N(\bar{a}+k\bar{\delta}_2;\bar{b}),$$

and by symmetry,

$$N(\bar{a};\bar{b}+k\bar{\delta}_1) \geq N(\bar{a};\bar{b}+k\bar{\delta}_2).$$

This latter inequality can be interpreted as saying that if $a_1 \leq a_2$ and $b_1 \leq b_2$ then it is as least as likely that a card of type 2 will be in the next k cards than one of type 1.

The final result in this section originated from the following conjecture: The probability that the next card is of type i cannot *decrease* if the next guess is type i and it is incorrect. In other words,

$$\frac{N(\bar{a};\bar{b}+\bar{\delta}_i)}{N(\bar{a};\bar{b})} \geq \frac{N(\bar{a};\bar{b}+2\bar{\delta}_i)}{N(\bar{a};\bar{b}+\bar{\delta}_i)}. \qquad (20)$$

This will turn out to be a consequence of the following more general result.

THEOREM 6. *For fixed $\bar{a}$ and $\bar{b}$, define the sequence $n_k \equiv N(\bar{a};\bar{b}+k\bar{\delta}_1)$. Then the sequence $\{n_k\}_{k\in\omega}$ is logarithmically concave, i.e.,*

$$n_k^2 \geq n_{k+1}n_{k-1} \qquad \text{for all } k.$$

Proof. It will be enough to show

$$n_1^2 \geq n_0 n_2. \tag{21}$$

Consider the matrix $M(\bar{a};\bar{b})$ written as shown in Fig. 3. We have permuted the columns so that the last b_1 guesses made so far are the b_1 guesses of type 1. The block B consists of the a_1 by 2 block adjoining the (new) a_1 by b_1 block of 0's (see Fig. 3). Let M_i denote the matrix formed from $M(\bar{a};\bar{b})$ by replacing the first i columns of 1's in B by 0's, $i = 0, 1, 2$. Then

$$M_0 = M(\bar{a};\bar{b}), \qquad M_1 = M(\bar{a};\bar{b} + \bar{\delta}_1), \qquad M_2 = M(\bar{a};\bar{b} + 2\bar{\delta}_2).$$

Let m_j denote the number of permutation choices of 1's from $M(\bar{a};\bar{b})$ having 1's in the first j columns of B and no 1's in the remaining $2 - j$ columns of B, $j = 0, 1, 2$. Thus

$$\begin{aligned} n_0 &= \operatorname{Per} M_0 = m_0 + 2m_1 + m_2, \\ n_1 &= \operatorname{Per} M_1 = m_0 + m_1, \\ n_2 &= \operatorname{Per} M_2 = m_0. \end{aligned}$$

A simple calculation shows that (21) is equivalent to

$$m_1^2 \geq m_0 m_2. \tag{21$'$}$$

To prove (21′) we first generalize the problem. Let M denote an n by n matrix of 0's and 1's having the structure shown in Fig. 4 (we have transposed the matrix for notational convenience later on). The top 2 rows of M are identical, the leftmost b columns of M are identical, and all entries of B are 1's. The submatrix M' is arbitrary. As before, let m_j denote the number of permutation choices of 1's which can be made in M so that 1's occur in B in exactly the first j rows, $j = 0, 1, 2$. We will show that (21′) actually holds in this more general case.

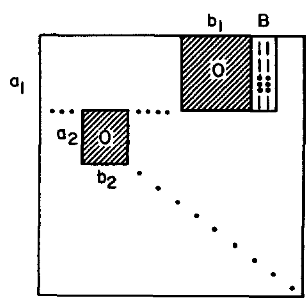

FIG. 3. A permuted form of $M(\bar{a},\bar{b})$.

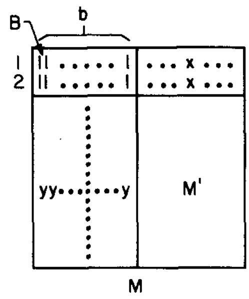

FIGURE 4

The matrix M can be viewed as a bipartite graph $G(M)$ in a natural way (see [7] for graph theory terminology). The vertex sets are $\{1,2,\ldots,n\}$ and $\{1',2',\ldots,n'\}$, corresponding to the row and column labels, respectively, of M. We have an edge $\{i,j'\}$ between i and j' *iff* the (i,j') entry $m_{i,j'}$ of M is 1. In this interpretation, a permutation choice of 1's in M is exactly a *matching* in $G(M)$, i.e., a set of n mutually disjoint edges. Let us call a matching an m_k-matching, $k = 0,1,2$, if it corresponds to a permutation choice having k 1's in the first k rows of B.

Consider an arbitrary m_2-matching μ_2 of $G(M)$. By definition, it must come from a permutation choice having 1's in both rows of B. Hence, μ_2 has edges $\{1,i'\}$ and $\{2,j'\}$ for some i', $j' \le b$, $i \neq j'$. Now let μ_0 be an arbitrary m_0-matching of $G(M)$. Consider the union $H = H(\mu_2,\mu_0)$ of the two graphs μ_2 and μ_0 (multiple edges are allowed). Since every vertex of H has degree 2 then H consists of a disjoint union of cycles. Denote the portions of the cycles containing $\{1,i'\}$ and $\{2,j'\}$ as shown in Fig. 5. Since μ_0 is an m_0-matching, all eight points, $u',1,i',v,x',2,j',y$ are distinct. By the regularity assumption of M, we know that the following are also edges of $G(M)$:

$$\{u',2\},\ \{2,i'\},\ \{x',1\},\ \{1,j'\},\ \{i',y\},\ \{j',v\}.$$

FIGURE 5

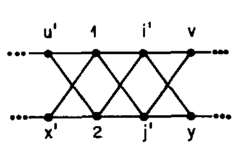

FIG. 6. A portion of $G(M)$.

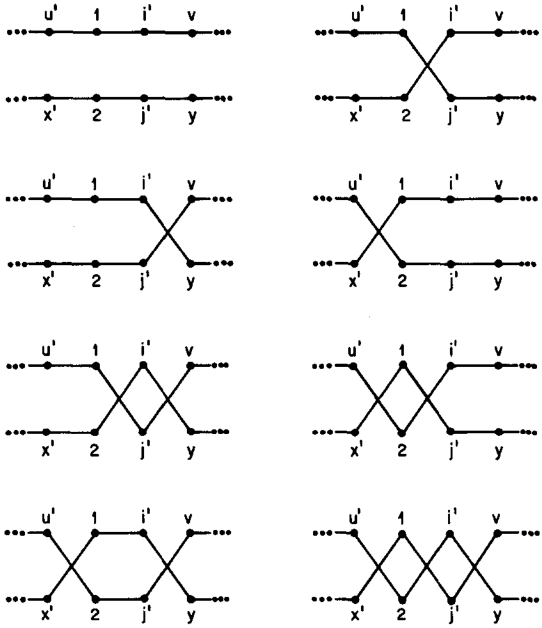

FIG. 7. Eight pairs (μ_2^*, μ_0^*) related to (μ_2, μ_0).

In Fig. 6 we show pictorially the various edges $G(M)$ must have in the cycle(s) containing 1 and 2.

The next step is to note that there are in fact *eight* different pairs (μ_2^*, μ_0^*) which differ from μ_2 and μ_0, respectively, only in edges between the points $u', 1, i', v, x', 2, j', y$. We illustrate these in Fig. 7. Note that in two of these, $\{1, i'\}$ and $\{2, j'\}$ are in different cycles and in two others, they are in a single cycle (see Fig. 7).

The key idea we now employ is to associate with the eight related pairs (μ_2^*, μ_0^*) eight other pairs $(\mu_1^*, \hat{\mu}_1^*)$. Each μ_1^* and $\hat{\mu}_1^*$ will be an m_1-matching; all the pairs $(\mu_1^*, \hat{\mu}_1^*)$ will be distinct. In Fig. 8 we show the basic decomposition patterns.

In each of (a), (b), (c), (d) all cycles have even length. Of course, portions of the graph not involved are always the same and are not shown. Thus, in each case we can decompose $\mu_2^* \cup \mu_0^*$ into two matchings μ_1^* and $\hat{\mu}_1^*$ (by choosing alternate edges) where the edges not in the cycle(s) containing 1 and 2 are partitioned in the same way they are in μ_2 and μ_0. Further, each of these matchings is an m_1-matching. The required *eight* pairs $(\mu_1^*, \hat{\mu}_1^*)$ come from taking both orders of the pairs formed above. It is not difficult to see that all eight such pairs are distinct. Moreover, a different choice of $\bar{\mu}_2, \bar{\mu}_0)$ will result in completely distinct pairs $(\bar{\mu}_1^*, \hat{\bar{\mu}}_1^*)$.

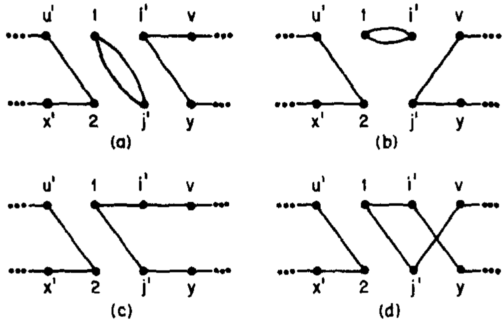

FIGURE 8

This injection of (sets of eight) pairs (μ_2, μ_0) into (sets of eight) pairs $(\mu_1, \hat{\mu}_1)$ proves (21′) and the proof of Theorem 6 is complete. □

An immediate consequence of Theorem 6 is the following.

COROLLARY. *The ratio*

$$\frac{N(\bar{a};\bar{b}+(k+1)\bar{\delta}_1)}{N(\bar{a};\bar{b}+k\bar{\delta}_1)}$$

is a nondecreasing function of $k \geq 0$.

The card interpretation of this is that each incorrect guess of type 1 makes it no less likely that the *next* card is of type 1.

It is tempting to conjecture that the following stronger result holds, namely, after an incorrect guess of type 1, the probability that the next card is of type 2 cannot increase. After all, how can knowing that card k is not of type 1 *increase* the probability that card $k+1$ is of type 2? Exactly how this can happen is shown in the following simple (and unexpected) example.

EXAMPLE. $n = 3$, $a_1 = a_2 = a_3 = 1$, $b_1 = b_2 = 0$, $b_3 = 1$. Then,

$$N(\bar{a};\bar{b}) = 4, \qquad N(\bar{a};\bar{b}+\bar{\delta}_2) = 3$$

$$N(\bar{a};\bar{b}+\bar{\delta}_1) = 3, \qquad N(\bar{a};\bar{b}+\bar{\delta}_1+\bar{\delta}_2) = 2.$$

Thus, after an incorrect guess "Card 1 = type 3", the probability that Card 2 is of type 1 is 1/4 and that it is of type 2 is also 1/4. However, after the second incorrect guess "Card 2 = type 1", the probability that Card 3 is of type 1 has increased to 1/3 (as we expect by the Corollary) but the probability that Card 3 is of type 2 has also increased to 1/3.

Concluding Remarks

Of course there are a large number of properties of $N(\bar{a};\bar{b})$ we have left untouched. Indeed the whole subject of permanents of matrices is a challenging area, where even the simplest looking questions can offer substantial resistance, e.g., the conjecture of van der Waerden (see [5]) that the permanent of any doubly stochastic n by n matrix is at least $n!/n^n$.

A question which seems to be in this category is the following. Given $\bar{a}$, for which $\bar{b}$ (with $b_1 + \cdots + b_r = n$ is $N(\bar{a};\bar{b})$ maximized? It has been shown by Efron [4] and follows from the Schur convexity demonstrated in Theorem 3, that for $\bar{a} = \bar{1}$, the choice $\bar{b} = \bar{1}$ maximizes $N(\bar{1};\bar{b})$. However, it is not unique since the choice $\bar{b} = (0,1,2)$ also satisfies $N(\bar{1};\bar{1}) = N(\bar{1};\bar{b}) = 2$.

Another question related to the card question which we have been unable to settle is the so-called "persistence" conjecture. This asserts that for any optimal guessing strategy (one which maximizes the expected number of correct guesses with "right–wrong" feedback), if type k is ever incorrectly guessed then the *next* guess should also be type k. In other words, the only time a new type should ever be guessed is immediately after a correct guess is made.

Appendix

Values of $N(\bar{a};\bar{b})$ for $n = 9$, $\bar{a} = (3,3,3)$

$\bar{b}$	$N(\bar{3};\bar{b})$	$\bar{b}$	$N(\bar{3};\bar{b})$
000	362,880	045	8,640
001	241,920	111	116,640
002	151,200	112	77,760
003	86,400	113	48,284
004	43,200	114	27,216
005	17,280	115	12,960
006	4,320	116	4,320
011	166,320	122	53,568
012	108,000	123	34,776
013	64,800	124	20,736
014	34,560	125	10,800
015	15,120	126	4,320
016	4,320	133	23,760
022	73,440	134	15,120
023	46,656	135	8,640
024	26,784	144	10,368
025	12,960	222	37,584

$\bar{b}$	$N(\bar{3}; \bar{b})$	$\bar{b}$	$N(\bar{3}; \bar{b})$
026	4,320	223	25,056
033	31,752	224	15,552
034	19,872	225	8,640
035	10,800	233	17,280
036	4,320	234	11,232
044	13,824	333	12,096

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