Roots of Ramsey theory

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1. Introduction

Among the problems appearing on the William Lowell Putnam Competition in 1953 was the following combinatorial question: "Six points are in general position in space. The fifteen line segments joining them in pairs are drawn, and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color." This problem stimulated the paper "Combinatorial relations and chromatic graphs" by Andy Gleason and Robert Greenwood [13], written later that year, which generalized the Putnam result in several directions. This paper is now recognized as a classic in the development of Ramsey theory, a branch of combinatorics which basically deals with partition-invariant structure. The essence of Ramsey theory is succinctly captured in the description of T. Motzkin: "Complete disorder is impossible." In this note we will describe various aspects of this subject, with special focus on its origins.

2. Ramsey's Theorem

Ramsey theory derives its name from the following result proved in 1928 by Frank Ramsey.

Theorem (Ramsey [26]). For all choices of positive integers k, l, r, there exists a least integer $R = R(k, l; r)$ so that for any partition of the k-element subsets of $[R] = \{1, 2, \dots, R\} = C_1 \cup \cdots \cup C_r$, there is an l-element set $X \subseteq [R]$ so that for some i, all the k-element subsets of X belong to C_i.

Ramsey had just graduated in 1925 as the top mathematics student at Cambridge University, and before his untimely death at the age of 26 in 1930, had made seminal contributions to economics, probability, decision theory and cognitive psychology [24].

The Putnam problem on 6 points is equivalent to the assertion that $R(2, 3; 2) = 6$, since the coloring assigning red to the pairs $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$, $\{5, 1\}$, and blue to the pairs $\{1, 3\}$, $\{2, 4\}$, $\{3, 5\}$, $\{4, 1\}$, $\{5, 2\}$ clearly has no monochromatic triangle.

For the case of $k = 2$, pairs from $[R]$ are usually identified with the edges of the complete graph on the set $[R]$. It is also traditional to call a partition into r sets $C_1 \cup \cdots \cup C_r$ an *r-coloring*, with C_i denoting the i^{th} color class, and sets contained in a single C_i *monochromatic*.

In the paper [13] of Gleason and Greenwood, the values $R(2, 4; 2) = 18$ and $R(2, 3; 3) = 17$ are established. It is interesting to note that in spite of extensive efforts over the past 37 years since [13] appeared, no other nontrivial values of $R(k, l; r)$ have been determined. The upper bounds for $R(2, 4; 2)$ and $R(2, 3; 3)$ are relatively straightforward. The lower bounds are both beautiful algebraic constructions. To show $R(2, 4; 2) > 17$, the pairs $\{i, j\}$ from $GF(17)$ are colored red if $i - j$ is a square in $GF(17)$, and blue otherwise. To show $R(2, 3; 3) > 16$, the pairs $\{i, j\}$ of $GF(16)$ are colored according to the cubic character of $i - j$ in $GF(16)$. This led the authors to suggest that in general, the extremal colorings for determining the values of $R(k, l; r)$ would depend on deep algebraic properties. However, this has yet to be confirmed since no other values are currently known!

In [13], Gleason and Greenwood also gave the estimates

$$41 < R(2,3;4) < 66.$$

The best bounds for this currently known are:

$$50 < R(2,3;4) < 65.$$

The lower bound is due to Fan Chung [3] in 1973 (who used ideas from cyclotomy); the upper bound was proved by Jon Folkman [8] in the late 1960's but was only published in 1974, five years after Folkman's tragic death.

3. The Erdös-Szekeres Theorem

It was the tradition among many of the young mathematicians in Budapest in the early 1930's to meet in parks and coffee houses to discuss all sorts of matters mathematical. It was during one of these gatherings which included Paul Erdös and George Szekeres that a curious discovery of one of the members of the circle, Esther Klein, emerged: Given any five points in the plane, some four form a convex quadrilateral. They soon made the general conjecture: For any n there exists N so that given N points in the plane, some n form a convex set. In a personal account of the social and mathematical climate of these times, Szekeres [4, preface] gives the following description:

> I have no clear recollection how the generalization actually came about; in the paper we attributed it to Esther, but she assures me that Paul had much more to do with it. We soon realized that a simple-minded argument would not do and there was a feeling of excitement that a new type of geometric problem emerged from our circle which we were only too eager to solve. For me, the fact that it came from Epszi (Paul's nickname for Esther, short for "epsilon") added a strong incentive to be the first with a solution and after a few weeks I was able to confront Paul with a triumphant 'E.P., open your wise mind'. What I had really found was Ramsey's Theorem, from which [the theorem] easily followed. Of course, at that time none of us knew about Ramsey.

At that time Erdös found an alternate proof based on what is now called the monotone subsequence theorem: any sequence of $n^2 + 1$ distinct integers must contain a *monotone* subsequence of length $n + 1$. This approach gives a much smaller upper bound for the least value $f(n)$ of N which assures the existence of a convex n-gon. Both proofs were included in the 1935 paper [6] of Erdös and Szekeres, which is largely responsible for the popularity of Ramsey theory in recent years. A long-standing conjecture is that $f(n) = 2^{n-2} + 1$, but this is only known to be true for $n \leq 5$.

Erdös sometimes calls this result the Happy End Theorem since George Szekeres and Esther Klein were subsequently married, and moved to Australia where they have lived and worked for the past 40 years.

3.1. van der Waerden's Theorem

At about the same time that Ramsey published his classic paper [26], B.L. van der Waerden had come across the following combinatorial question, which seems to have originated with I. Schur in the early 1920's (although it also has been attributed to a Dutch mathematician Baudet):

> Is it true that in any 2-coloring of the positive integers, there always exist monochromatic k-term arithmetic progressions for every k?

van der Waerden [37] describes what happened after he mentioned the conjecture to Emil Artin and Otto Schreier at lunch one day:

> After lunch we went into Artin's office in the Mathematics Department of the University of Hamburg, and tried to find a proof. We drew some diagrams on the blackboard. We had what the Germans call *Einfälle:* sudden ideas that flash into one's mind. Several times such new ideas gave the discussion a new turn, and one of the ideas finally led to the solution.

What van der Waerden actually proved in his 1927 paper [36] was this:

> For any k and r there is a least $W(k,r)$ such that if $[W(k,r)] = C_1 \cup \cdots \cup C_r$ then some C_i contains a k-term arithmetic progression.

Until very recently, all known proofs of van der Waerden's Theorem employed a double induction argument, resulting in upper bounds on $W(k,r)$ which grew like the Ackermann function, and, in particular, were not primitive recursive. However, in 1988 Shelah [30] found a different proof which in particular yielded the following bound for $W(k) := W(k,2)$:

$$W(k) \le \left.\begin{matrix} 2^{2^{\cdot^{\cdot^{2}}}} \\ \vdots \\ 2^{2^{\cdot^{\cdot^{2}}}} \\ 2^{2^{\cdot^{\cdot^{2}}}} \\ 2^{2^{2^{2}}} \end{matrix}\right\} k \text{ layers}$$

Thus,

$$W(3) \le \left. 2^{2^{\cdot^{\cdot^{2}}}} \right\} 65536 \text{ 2's}.$$

The only known values of $W(k)$ are

$$W(2) = 3, \qquad W(3) = 9, \qquad W(4) = 35, \qquad W(5) = 178.$$

The best available lower bounds for $W(k)$ are on the order of 2^k. The author has had the following offer out for some years:

Conjecture 3.1 (\$1000). For all k,

$$W(k) \le \left. 2^{2^{\cdot^{\cdot^{2}}}} \right\} k \text{ 2's}.$$

No doubt, the truth is much closer to 2^k than the above.

3.2. Szemerédi's Theorem

Soon after the appearance of van der Waerden's paper in 1927, Erdös and Turán raised the question as to *which* C_i must contain the desired arithmetic progressions. In particular, they proposed the following conjecture.

For $X \subseteq \mathbb{N} = \{1, 2, 3, \ldots\}$, define $\overline{d}(X)$, the upper density of X, by

$$\overline{d}(X) := \overline{\lim_{N\to\infty}} \frac{|X \cap \{1, 2, \ldots, N\}|}{N}.$$

Conjecture 3.2 (Erdös and Turán [7]). If $\overline{d}(X) > 0$ then X contains k-term arithmetic progressions for all k. (Note that this would clearly imply van der Waerden's theorem.)

The first significant advance was made by K.F. Roth in 1954 [27] who established the conjecture for $k = 3$. This was followed in 1969 by a proof for $k = 4$ by Szemerédi [33], and finally, in 1975, a proof by him for general k. Szemerédi's proof [34] was a marvel of combinatorial ingenuity and introduced his well known regularity lemma, which has had wide application in combinatorics in recent years. (This result also was the first \$1000 problem of Erdös to be solved.)

3.3. Ergodic theory

Within the past 15 years, a quite different approach has been developed by Furstenberg, Katznelson, Ornstein, Weiss, and others, using methods from ergodic theory and topological dynamics. The setting is basically the following. A dynamical system (X, T) consists of a compact metric space X together with a continuous map $T : X \to X$. A point $\alpha \in X$ is said to be recurrent if there exist $n_1 < n_2 < \cdots$ such that

$$T^{n_k}(\alpha) \to \alpha \quad \text{as} \quad k \to \infty.$$

The fundamental recurrence theorem of G.D. Birkhoff from 1912 asserts that any (compact) dynamical system has recurrent points. This was generalized by Furstenberg and Weiss in 1978, who proved:

Theorem (Multiple Birkhoff Recurrence Theorem; see [9]). If X is a compact metric space and $T_j : X \to X$, $1 \le j \le m$, are commuting continuous maps then there exists $\alpha \in X$ and $n_1 < n_2 < \cdots$ such that $T_1^{n_k}(\alpha) \to \alpha$, $T_2^{n_k}(\alpha) \to \alpha, \ldots, T_m^{n_k}(\alpha) \to \alpha$ as $k \to \infty$.

They show that by taking X to be the set of all r-colorings C of $\mathbb{N}$ with the metric ρ given by

$$\rho(C, C') = \frac{1}{n}$$

where n is the least integer where the colorings C and C' differ, and the T_j are various "shift" operators of the type $(TC)(i) = C(i + 1)$, van der Waerden's theorem follows as an immediate corollary.

In a similar vein, the Poincaré recurrence theorem asserts that if $(X, \mathcal{B}, \mu)$ is a measure space and T is a measure-preserving transformation on X then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \ge 1$ with

$$\mu(A \cap T^{-n}(A)) > 0.$$

This was generalized by Furstenberg in 1977 as follows:

Theorem (Multiple Poincaré Recurrence Theorem [9]). If $(X, \mathcal{B}, \mu)$ is a measure space and $T_1, T_2, \ldots, T_m$ are commuting measure-preserving transformations on X then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \geq 1$ with

$$\mu(A) \cap T_1^{-n}(A) \cap T_2^{-n}(A) \cap \cdots \cap T_m^{-n}(A) > 0.$$

Furstenberg then shows [9, 10, 12] that Szemerédi's theorem follows as an immediate corollary. As we will see in the next section, this approach has yielded significant results which so far have been unobtainable by any other methods, and provides an unexpected but most satisfying link between combinatorics and ergodic theory.

3.4. The Hales-Jewett Theorem

The combinatorial core of van der Waerden's theorem is really a statement concerning certain "line-like" structures in Cartesian products of finite sets, as was first made clear in the work of Hales and Jewett in 1963. In particular, with $A = \{a_1, a_2, \ldots, a_t\}$, define a (combinatorial) *line* L in $A^N = \{(x_1, \ldots, x_N) : x_i \in A\}$ to be a set of t points $p(k) = (p_1(k), p_2(k), \ldots, p_N(k))$, $1 \leq k \leq t$, where for some nonempty set of indices $I \subseteq \{1, 2, \ldots, N\}$,

$$p_i(k) = a_k \qquad \text{for } i \in I,$$

and

$$p_i(k) = b_i \in A \qquad \text{for } i \notin I.$$

Thus, L is just the set of points satisfying the set of equations $x_{i_1} = x_{i_2} = \cdots = x_{i_r}$, $I = \{i_1, i_2, \ldots, i_r\}$, and $x_j = b_j \in A$, $j \notin I$. Then we have the following:

Theorem (Hales-Jewett (1963) [21]). For any A and r, there exists $N = N(A, r)$ so that if $A^N = C_1 \cup \cdots \cup C_r$ then some C_i contains a line.

To see that this implies van der Waerden's theorem, simply take $A = \{1, 2, \ldots, t\}$ and let $N = N(A, t)$. For any r-coloring of $[tN] = \{1, 2, \ldots, tN\} = C_1 \cup \cdots \cup C_r$, we can define an induced r-coloring of $A^N = C'_1 \cup \cdots \cup C'_N$ by setting:

$$(x_1, x_2, \ldots, x_N) \in C'_i \iff x_1 + x_2 + \cdots + x_N \in C_i.$$

By Hales-Jewett, there is a monochromatic line in this coloring which clearly corresponds to a monochromatic t-term arithmetic progression in the original coloring of $[tN]$.

We remark that a higher-dimensional version of the Hales-Jewett theorem was found in 1971 by the author and Bruce Rothschild [17]. This extremely general result has stimulated a new wave of Ramsey-like theorems in the past few years, including a proof with K. Leeb [15] of Rota's conjectured analogue of Ramsey's theorem for finite-dimensional vector spaces over finite fields.

It should be noted that while van der Waerden's theorem and the Hales-Jewett theorem are *partition* theorems, Szemerédi's theorem is of a different nature, being a stronger *density* theorem. It is natural to ask if there is density version of the Hales-Jewett theorem. That this does indeed exist has only very recently been established by Furstenberg and Katznelson, who proved:

Theorem (Furstenberg and Katznelson [11]). For all A and $\epsilon > 0$, if $N = N(A, \epsilon)$ is sufficiently large then any $X \subseteq A^N$ with $|X| > \epsilon|A|^N$ contains a line.

3.5. Schur, Rado and Hilbert

One of the earliest theorems of Ramsey type was proved by I. Schur in 1916 in connection with his work on modular versions of Fermat's Last Theorem. In particular he showed [29] that in any partition of $\mathbb{N} = C_1 \cup \cdots \cup C_r$ the equation $x + y = z$ can always be solved within a single C_i (i.e., $x + y = z$ has a monochromatic solution). This result follows easily from Ramsey's theorem — given an r-coloring of $\mathbb{N}$, form the induced r-coloring of the pairs $\{i, j\}$, $i > j$, by the map $i - j \leftrightarrow \{i, j\}$. A monochromatic triangle $i > j > k$ then produces the desired monochromatic numbers $i-j$, $j-k$ and $i-k$ since $(i - j) + (j - k) = i - k$.

This approach was greatly expanded by Schur's student, the late Richard Rado, whose dissertation in 1933 contained many beautiful results of this type. A particularly nice one is the following. Call a linear equation

$$E(x_1, \ldots, x_n) = \sum_i a_i x_i = 0,$$

with a_i nonzero integers, *partition regular* if $E(x_1, \ldots, x_n)$ always has monochromatic solutions whenever $\mathbb{N}$ is colored with finitely many colors.

Theorem (Rado [25]). $E(x_1, \ldots, x_n)$ is partition regular if and only if $E(\epsilon_1, \ldots, \epsilon_m) = 0$ for some choice of $\epsilon_i = 0$ or 1, not all 0.

It is not hard to show that there is a density version of this result:

Theorem (Graham [14]). $E(x_1, \ldots, x_n)$ has a nontrivial solution in any set of positive upper density if and only if $E(1, 1, \ldots, 1) = 0$.

Thus, $x_1 + x_2 = 2x_3$ is density regular (Roth's theorem for 3-term arithmetic progressions), while $x_1 + x_2 = x_3$ is not (since it has no solution in the set of odd integers, for example), although it is partition regular. On the other hand, $x_1 + x_2 = 3x_3$ is not even partition regular.

Rado [25] also characterizes partition regular *systems* of linear equations. This implies the following generalization of Schur's result, which was also proved independently (and later) by Folkman [8] and Sanders [28].

Theorem (Finite Sums Theorem). For any n and r, if $\mathbb{N} = C_1 \cup \cdots \cup C_r$ then there exist $x_1, \ldots, x_n$ such that all nonempty subset sums $x_{i_1} + \cdots x_{i_s}$ have the same color.

Remarkably, a similar (though weaker) result of this type had already been proved by Hilbert in 1892, in connection with his celebrated irreducibility theorem (which asserts that if $P(X, Y) \in \mathbb{Z}[X, Y]$ is an irreducible polynomial then $P(a, Y) \in \mathbb{Z}[Y]$ is irreducible for some $a \in \mathbb{Z}$).

Theorem (Hilbert's Cube Lemma [22]). For any n and r, if $N = C_1 \cup \cdots \cup C_r$ then there exist a and $x_1, \ldots, x_n$ such that all sums $a + x_{i_1} + \cdots + x_{i_s}$ (i.e., all subset sums of the x_i translated by a) have the same color.

In this sense, Hilbert could properly be identified as having proved the first result in Ramsey theory. We remark that in 1974, Hindman proved a striking generalization of the Finite Sums Theorem.

Theorem (Hindman [23]). If $\mathbb{N} = C_1 \cup \cdots \cup C_r$ then there exists an *infinite* set $X \subset \mathbb{N}$ such that all nonempty finite subset sums from X have the same color.

This important result has formed the foundation for much recent work in combinatorial set theory [2].

3.6. Ramsey numbers again

Since it appears hopeless at present to obtain any further exact values of the Ramsey function $R(k, l; r)$ beyond those found by Gleason and Greenwood, researchers have looked at alternatives. One approach is to study the so-called asymmetric Ramsey number $r(k, l)$, defined to be the least integer r such that any red-blue coloring of the edges of the complete graph K_r or r vertices contains either a red K_k or a blue K_l. The known values (and some bounds) are shown in Table 3.1.

	l = 3	4	5	6	7	8	9	10
k = 3	6	9	14	18	23	28	36	40–43
4	18	25–27	34–43					
5		43–52						

Table 3.1 $r(k, l)$

While the coloring showing $r(3, 9) > 35$ has cyclic symmetry, those for $r(3, 6)$, $r(3, 7)$ and $r(3, 8)$ are rather unsymmetrical [20].

In another direction, people have investigated the asymptotic behavior of $r(n) := r(n, n)$ as n goes to infinity. The major step along these lines was taken by Erdös [4] in 1947. It was also in this paper that he introduced his use of the so-called probabilistic method, a powerful technique for proving the existence of desired objects without having (or being able) to construct them (see [1] for a comprehensive discussion). In particular, Erdös proved that

$$c_1 n 2^{n/2} < r(n) < \frac{4^n}{n^{c_2}}$$

for certain constants $c_1, c_2 > 0$. Since this time, the only improvements have been in the value of the c_i's. The best lower bound known is due to Spencer [31] (who was a student of Andy Gleason) while the best upper bound available is due to Thomason [35]. Erdös offers \$250 for showing that

$$\lim_{n \to \infty} r(n)^{1/n}$$

exists, and \$500 for finding it, if it does exist.

Space limitations do not permit us to discuss any more of the many exciting directions currently being pursued by researchers in the field. Clearly, much remains to be done. For fuller discussions of many of the topics covered (and more that weren't), the reader is referred to [2, 5, 14, 18, 19, 16, 32].

Final note

After the May 4 celebration I received the note following the bibliography from Andy. I'm pleased to announce here that there may soon be answers to some of the open questions in Ramsey theory—perhaps a further flowering from these roots.

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