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On Schur Properties of Random Subsets of Integers*

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A classic result of I. Schur [9] asserts that for every $r \geqslant 2$ and for n sufficiently large, if the set $[n] = \{1, 2, ..., n\}$ is partitioned into r classes, then at least one of the classes contains a solution to the equation $x + y = z$. Any such solution with $x \neq y$ will be called a *Schur triple*. Let us say that $A \subseteq [n]$ has the *Schur property* if for every partition (or 2-coloring) of $A = R \cup B$ (for *red* and *blue*), there must always be formed some *monochromatic* Schur triple, i.e., belonging entirely to either R or B. Our goal here is to show that $n^{1/2}$ is a threshold for the Schur property in the following sense: for any $\omega = \omega(n) \to \infty$, almost all sets $A \subseteq [n]$ with $|A| < n^{1/2}/\omega$ do not possess the Schur property, while almost all $A \subseteq [n]$ with $|A| > \omega n^{1/2}$ do.

1. INTRODUCTION

A classic result of I. Schur [9] asserts that for every $r \geqslant 2$ and for n sufficiently large, if the set $[n] = \{1, 2, ..., n\}$ is partitioned into r classes, then at least one of the classes contains a solution to the equation $x + y = z$. Any such solution with $x \neq y$ will be called a *Schur triple*. Let us say that $A \subseteq [n]$ has the *Schur property* if for every partition (or 2-coloring) of $A = R \cup B$ (for *red* and *blue*), there must always be formed some *monochromatic* Schur triple, i.e., belonging to either R or B. Our goal here is to show that $n^{1/2}$ is a threshold for the Schur property in the following sense: for any $\omega = \omega(n) \to \infty$, almost all sets $A \subseteq [n]$ with $|A| < n^{1/2}/\omega$ do not possess the Schur property, while almost all $A \subseteq [n]$ with $|A| > \omega n^{1/2}$ do. (In fact, we will prove somewhat sharper results than this; see Theorem 1 and Theorem 2.)

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The existence of a threshold is guaranteed by a general result of Bollobás and Thomason [1] dealing with monotone properties. The fact that $n^{1/2}$ is the threshold can be intuitively explained by the observation that in a random subset of size $n^{1/2}$, a typical element is contained in just a constant number of Schur triples. This may be easily verified by switching to another (but asymptotically equivalent) random model, where each element is included in a random set independently with some probability p (our notation for such a random set will be $[n]_p$). In fact, we will prove here the following stronger results:

THEOREM 1. *There exists a constant* $\alpha > 0$ *so that with probability tending to* 1 *as* $n \to \infty$, *every* 2-*coloring of* $[n]_p$, *with* $p = \omega(n)\, n^{-1/2}$, *where* $\omega(n) \to \infty$ *arbitrarily slowly*, *results in at least* $\alpha n^2 p^3$ *monochromatic Schur triples.*

THEOREM 2. *There exists a constant* $c > 0$ *so that with probability tending to* 1 *as* $n \to \infty$, *there exists a* 2-*coloring of* $[n]_p$, *with* $p = c/n^{1/2}$, *without any monochromatic Schur triple.*

Note that since there are essentially $n^2/4$ Schur triples in $[n]$, our Theorem 1 promises that a fixed fraction of all the Schur triples in $[n]_p$ will be monochromatic. In fact, this behavior has been shown to occur for $p = 1$ and any r-coloring of $[n]$, r fixed, by Frankl, Graham and Rödl [2].

Our theorems resemble a result about 3-term arithmetic progressions (called further *arithmetic triples*), which is a very special case of a general threshold for all density-partition regular systems of homogeneous linear equations found in [8] (see also [7] and [5]).

THEOREM 3. *There exist constants* $C > 0$ *and* $\alpha > 0$ *so that with probability tending to* 1 *as* $n \to \infty$, *every* 2-*coloring of* $[n]_p$, *with* $p = C/\sqrt{n}$, *results in at least* $\alpha n^2 p^3$ *monochromatic arithmetic triples.*

THEOREM 4. *There exists a constant* $c > 0$ *so that with probability tending to* 1 *as* $n \to \infty$, *there exists a* 2-*coloring of* $[n]_p$, *with* $p = c/n^{1/2}$, *without any monochromatic arithmetic triple.*

Indeed, the proofs of Theorems 2 and 4 are identical. We will see, however (cf. discussion at the beginning of Section 3), that the proof of Theorem 3 cannot be carried through for Theorem 1.

2. THE DETERMINISTIC CASE

Our proof of Theorem 1 is based on an approach of Goodman [3] for a deterministic question about graphs. Rather than counting monochromatic triangles directly, Goodman instead bounded from above the number of triangles in which two colors appear. We will follow the same idea here.

Before we outline the analogous argument for Schur triples, which will subsequently be adapted to a probabilistic setting, we will first show how the Goodman result (but not his method) can be directly utilized for the deterministic version of our problem.

Goodman proved that in every 2-coloring of the edges of a complete graph on n vertices, there are at least $n^3/24+O(n^2)$ monochromatic triangles formed. A natural connection between triangles and Schur triples on the same vertex $[n]$ yields a lower bound for the number of monochromatic Schur triples. Namely, to each triangle $\{i<j<k\}$ we can associate the Schur triple $\{j-i, k-j, k-i\}$ (provided $j-i \neq k-j$). Thus, each Schur triple has at most $2n$ triangles associated to it, so that Goodman's result implies that in any 2-coloring of $[n]$ there are at least $(1+o(1))\, n^2/48$ monochromatic Schur triples. This argument can be sharpened slightly by observing that the Schur triple $\{a, b, c\}$ with $a+b=c$ can only be associated to at most $2(n-c)$ triangles. This implies that there must always be at least $n^2/38$ monochromatic Schur triples (more precisely, at least $(1+o(1))\, \alpha n^2$ monochromatic Schur triples where $\alpha=0.026626...$).

In the other direction, applying a random 2-coloring to $[n]$ gives $n^2/16$ as an upper bound for the number of monochromatic Schur triples a 2-coloring of $[n]$ must contain. The best construction we know of (due to Dorm Zeilberger [11]) has only $n^2/22$ monochromatic Schur triples. This comes from the 2-coloring h of $[n]$ defined by:

$$h(x)=\begin{cases} \text{Red} & \text{if} \quad 4n/11 \leqslant x < 10n/11, \\ \text{Blue} & \text{otherwise.} \end{cases}$$

It would be interesting to know what the best value of the constant actually is for this problem.

We now give an argument (based on Goodman's approach) which, although guarantees only $n^2/100$ monochromatic Schur triples in every 2-coloring, will serve as a framework for the proof of the probabilistic version.

Let T denote the total number of Schur triples in $[n]$, so that $T \sim (1/4)\, n^2$. For each 2-coloring of $[n]=B \cup R$, let M, A and P denote the numbers of monochromatic Schur triples, achromatic Schur triples and achromatic pairs, respectively, which are formed. Thus,

$$A=T-M, \qquad P=|B|\,|R|. \tag{1}$$

We first claim that

$$A \leqslant P. \tag{2}$$

To see this, we simply count the number N of pairs (p, t) where p is an achromatic pair contained in the (achromatic) Schur triple t, in two ways. On one hand, every achromatic triple contains two achromatic pairs, so $N = 2A$. On the other hand, every achromatic pair $\{x < y\}$ can be extended to at most two achromatic Schur triples $\{x, y-x, y\}$ and $\{x, y, x+y\}$. (Sometimes, $y - x = x$ or $x + y > n$, and neither of them counts.) Thus, $N \leqslant 2P$ and (2) follows.

Assume first that the coloring is "unbalanced," i.e., for $\beta = 0.1$, $|B|\,|R| \leqslant ((1/4) - \beta^2)\, n^2$, or equivalently, say, $|R| \geqslant ((1/2) + \beta)\, n$. Then

$$M = T - A \geqslant T - P = T - |B|\,|R| = T - (\tfrac{1}{4} - \beta^2)\, n^2 \sim \beta^2 n^2 \geqslant 0.01 n^2 \quad (3)$$

as desired.

So, we can now assume instead that

$$|R| \leqslant (\tfrac{1}{2} + 0.1)\, n, \qquad |B| \geqslant (\tfrac{1}{2} - 0.1)\, n \quad (4)$$

hold. Set

$$|\{x \in B\colon x > n/2\}| = \lambda n.$$

Note that there are at least $\lambda n((1/2) - \lambda)\, n$ achromatic pairs (with both terms $> n/2$) which each have only one extension to a Schur triple. Hence, we can refine our previous estimate to obtain

$$2A \leqslant 2P - \lambda(\tfrac{1}{2} - \lambda)\, n^2 \leqslant 2[\tfrac{1}{4} - \tfrac{1}{2}\lambda(\tfrac{1}{2} - \lambda)]\, n^2. \quad (5)$$

If $1/20 \leqslant \lambda \leqslant 9/20$ then (5) implies $M > 9/800 > 0.01 n^2$ and we are done. Otherwise, one of the colors occurs on at least 70% of the elements in $[n/2]$. In this case we can apply the argument from the first (unbalanced) case with $\beta = 0.2$, to get to show that $M \geqslant 0.04(n/2)^2 = 0.01 n^2$ in this case as well.

3. THE PROBABILISTIC CASE

Although Theorem 1 is similar to Theorem 3, there is a significant difference between the two which forces us to alter the proof as compared to the one used in [8]. We think it may help to understand our proof if we first describe the proof which works for arithmetic triples and explain why it does not work for Schur triples.

In the proof of Theorem 3 one applies the classic two-round exposure technique, setting $p = p_1 + p_2$ and $p_1 \ll p_2$. In round 1 one generates the random subset $[n]_{p_1}$ and lets it be colored by an adversary in one of about 2^{np_1} ways. Then a subset D of $[n] \setminus [n]_{p_1}$ is determined, the elements of which form an arithmetic triple together with a red pair of elements of $[n]_{p_1}$ (say, red dominates in the coloring). Since there are at least

$\Theta(n^2p_1^2)=\Theta(n)$ such pairs, one can show that also $|D|=\Theta(n)$, and, by Szemerédi's Theorem ([10]), D, as a dense subset of $[n]$, contains $\Theta(n^2)$ arithmetic triples. The probability that none of them survive during the second round is, by a correlation inequality from [4], less than $e^{-\Theta(np_2)}$ which dominates the factor 2^{np_1}. Once there is an arithmetic triple in D_{p_2}, no matter how the adversary completes the coloring, a monochromatic triple is bound to appear. With an extra twist one can squeeze out of this proof not just one but in fact $\Theta(n^2p^3)$ monochromatic arithmetic triples.

Unfortunately, the Schur equation is a non-density partition regular equation, i.e., not every dense subset of $[n]$ contains a Schur triple. This is why we have to abandon the above approach and instead try to apply Goodman's idea of bounding the number of achromatic triples from above. As a consequence, however, we cannot use the classic 2-round exposure, since whatever upper bound we set after round 1, it can be exceeded as a result of round 2. This led us to the version of the 2-round exposure technique utilized in the proof of our Main Lemma.

To prove Theorem 1, we first need to establish several preliminary results. We will find it convenient to partition $[n]$ into k classes, $[n]=N_1\cup N_2\cup\cdots\cup N_k$, where k is a fixed large integer to be determined later and the partition enjoys certain quasi-random properties. Let t_i denote the number of Schur triples in N_i and let t'_i denote the number of Schur triples in $N'_i=N_i\cap[n/2]$, where $[n/2]=\{1,2,\ldots,\lfloor n/2\rfloor\}$. Furthermore, let t_{ij} denote the number of Schur triples with one element in N_i and two elements in N_j.

For each $v\in[n]$, and $1\leqslant i, j\leqslant k$, let $x^-(v,i,j)$ denote the number of Schur triples $\{v,x,y\}$ with $x\in N_i$, $y\in N_j$ and $v=x-y$, $y\neq v$. Similarly, let $x^+(v,i,j)$ denote the number of Schur triples $\{v,x,y\}$ with $x\in N_i$, $y\in N_j$ and $v=x+y$. Note that while in general $x^-(v,i,j)\neq x^-(v,j,i)$, it is always true that $x^+(v,i,j)=x^+(v,j,i)$. Finally, let $x^{+'}(v,i,j)$ and $x^{+''}(v,i,j)$ stand for the corresponding numbers with $x\in N'_i$, $y\in N'_j$, and $x\in N'_i$, $y\in N''_j=N_j\backslash[n/2]$, respectively. Observe that $x^{+'}(v,i,j)=x^+(v,i,j)$ for $v\leqslant n/2$.

For the next lemma and throughout the paper we adopt the notation $a_n\sim b_n$ to stand for the fact that for every $\varepsilon>0$ there is an integer n_0 such that for all $n>n_0$, $|a_n-b_n|<\varepsilon b_n$.

LEMMA 1. *For every integer $k\geqslant 2$ and n sufficiently large there exists a partition of $[n]=N_1\cup N_2\cup\cdots\cup N_k$ such that*:

(i) *For $1\leqslant i\leqslant k$,*

$$|N_i|\sim\frac{n}{k},\qquad |N'_i|\sim\frac{n}{2k}.$$

(ii) *For* $1 \leqslant i \leqslant k$,

$$t_i \sim \frac{n^2}{4k^3}, \qquad t'_i \sim \frac{n^2}{16k^3}, \qquad t_{ij} \sim \frac{3n^2}{4k^3}.$$

(iii) *Let* $\omega(n) \to \infty$ *arbitrarily slowly. Then,*

(a) *for all* $1 \leqslant v \leqslant n - \omega$ *and all* $1 \leqslant i, j \leqslant k$,

$$x^-(v, i, j) \sim \frac{n-v}{k^2};$$

(b) *for all* $\omega \leqslant v \leqslant n$ *and all* $1 \leqslant i < j \leqslant k$,

$$x^+(v, i, j) \sim \frac{v}{k^2},$$

while

$$x^+(v, i, i) \sim \frac{v}{2k^2};$$

(c) *for all* $n/2 \leqslant v \leqslant n - \omega$ *and all* $1 \leqslant i \neq j \leqslant k$,

$$x^{+'}(v, i, j) \sim \frac{n-v}{k^2},$$

while

$$x^{+'}(v, i, i) \sim \frac{n-v}{2k^2};$$

(d) *for all* $n/2 + \omega \leqslant v \leqslant n$ *and all* $1 \leqslant i, j \leqslant k$,

$$x^{+''}(v, i, j) \sim \frac{v - n/2}{k^2}.$$

Proof. We choose the classes N_i by flipping a fair "k-sided coin," i.e., each $v \in [n]$ is independently assigned to one of the N_i with equal probability $1/k$.

Define

$$I_i(v) = \begin{cases} 1 & \text{if} \quad v \in N_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$|N_i| = \sum_{v \in [n]} I_i(v)$$

has binomial distribution with expectation n/k and (i) follows by a standard application of Chebyshev's or Chernoff's inequality. In the remainder of the proof let upper case letters T and X, with appropriate subscripts and superscripts designate random variables corresponding to the numerical parameters denoted by the lower case counterparts.

To prove (ii), we apply Chebyshev's inequality to the random variable T_i. We can write T_i as

$$T_i = \sum_t T_i(t)$$

where t ranges over all Schur triples in $[n]$ and

$$T_i(t) = \begin{cases} 1 & \text{if} \quad t \subset N_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$\mathbb{E}(T_i) = \sum_t \mathbb{E}(T_i(t)) \sim \frac{n^2}{4k^3}. \tag{6}$$

and

$$\mathrm{Var}(T_i) = \sum_{t,\, t'} \mathrm{Cov}(T_i(t), T_i(t')) = \sum_{t \cap t' \neq \phi} \mathrm{Cov}(T_i(t), T_i(t')). \tag{7}$$

However, there are at most $O(n^3)$ pairs t, t' of Schur triples which intersect. Thus, by Chebyshev's inequality,

$$\mathbb{P}[|T_i - \mathbb{E}[T_i]| > \varepsilon \mathbb{E}[T_i]] < \frac{\mathrm{Var}(T_i)}{\varepsilon^2 \mathbb{E}(T_i)^2} = O(1/n) = o(1) \tag{8}$$

as $n \to \infty$ and the first part of (ii) follows. The other two parts of (ii) are proved in the same way, and so (ii) holds.

We shall only prove part (a) of (iii), since the other parts follow the same lines and, in fact, are even simpler since all the random variables with superscript "+" have binomial distributions. Observe that for each v the pairs $\{x, y\}$ with $v = x - y$, viewed as a graph, form a set of vertex-disjoint paths (the vertex sets of these paths are arithmetic progressions with difference v, beginning at elements $1, 2\ldots, v-1$) and therefore can be partitioned into two sets P and P', with $|P| = \lfloor (n-v)/2 \rfloor$ and $|P'| = \lceil (n-v)/2 \rceil$, so that all the pairs in P (and also P') are disjoint. Thus, $X^-(v, i, j)$ is a sum of two binomial distributions $B(1/2(n-v), 1/k^2)$. By the Chernoff inequality (see also Lemma 4 below), the probability that either of the binomial ingredients of $X^-(v, i, j)$ deviates from its mean by an ε-fraction is bounded by $4\exp(-\varepsilon^2(n-v)/6k^2)$. Hence, the probability that it happens for at least one v, $1 \leqslant v \leqslant n-\omega$, is bounded by

$$4 \sum_{1 \leqslant v \leqslant n-\omega} \exp\left(-\frac{\varepsilon^2(n-v)}{6k^2}\right),$$

which, as a remainder of geometric series, converges to 0, since $\omega \to \infty$. Summarizing, all three properties (i)–(iii) hold each with probability close to 1, and hence their intersection is nonempty. Thus, there exists a partition which satisfies all of them. ∎

So, let $[n] = N_1 \cup N_2 \cup \cdots \cup N_k$ be a fixed partition of $[n]$ satisfying (i)–(iii) in Lemma 1. For $p > \omega(n)\, n^{-1/2}$, $\omega(n) \to \infty$, let us sample $[n]$ with probability p. That is, we independently select each $v \in [n]$ with probability p to form $[n]_p = (N_1)_p \cup (N_2)_p \cup \cdots \cup (N_k)_p$ where $(N_i)_p$ denotes the induced subset of N_i. Let $T_{i,p}$, $T'_{i,p}$, and $T_{ij,p}$ designate random variables in this probability space, which correspond to previously defined numerical parameters t_i, t'_i, and t_{ij}. Finally, using $*$ to represent any of the superscripts $-$, $+$, $+'$ and $+''$, let $X^*_{pp}(v, i, j)$ correspond to the previously defined $x^*(v, i, j)$ and $X^*_{ppp}(v, i, j) = X^*_{pp}(v, i, j)$ if $v \in [n]_p$ and 0 otherwise. So, for example, $X^+_{pp}(v, i, j)$ stands for the number of Schur triples $\{x, y, v\}$ with $x \in (N_i)_p$, $y \in (N_j)_p$ and $v = x + y$, while $X^*_{ppp}(v, i, j)$ counts them only when, in addition, $v \in [n]_p$. Observe that, for each $* = +$, $+'$, $+''$, $X^*_{pp}(v, i, j)$ is a binomially distributed random variable with expectation $x^*(v, i, j)\, p^2$, while $X^-_{pp}(v, i, j)$, similar to $X^-(v, i, j)$ in the proof of Lemma 1, is a sum of two binomial random variables.

LEMMA 2. *With probability tending to* 1 *as* $n \to \infty$,

(i) *For all* $1 \leqslant i \leqslant k$,

$$|(N_i)_p| \sim \frac{np}{k}, \qquad |(N'_i)_p| \sim \frac{np}{2k}.$$

(ii) *For all* $1 \leqslant i \leqslant k$,

$$T_{i,p} \sim \frac{n^2p^3}{4k^3}, \qquad T'_{i,p} \sim \frac{n^2p^3}{16k^3}, \qquad T_{ij,p} \sim \frac{3n^2p^3}{4k^3}.$$

(iii) *For all* $1 \leqslant i, j \leqslant k$, *setting* $\mu = 2np^2/k^2 = 2\omega^2/k^2$,

$$D^*_{pp} = D^*_{pp}(i, j) = \{v\colon X^*_{pp}(v, i, j) \leqslant 2\mu\},$$

and

$$D^*_{ppp} = D^*_{ppp}(i, j) = \{v\colon X^*_{ppp}(v, i, j) \leqslant 2\mu\},$$

we have

(a) $\sum_{v \notin D^*_{pp}} X^*_{pp}(v, i, j) = o(n)$ *and*
(b) $\sum_{v \notin D^*_{ppp}} X^*_{ppp}(v, i, j) = o(np)$.

Proof. Part (i) is trivial. To see that, again, part (ii) follows by Chebyshev's inequality, observe that there are $O(n^3)$ pairs of Schur triples intersecting in one element and $O(n^2)$ pairs intersecting in two. They contribute to the variance of each of the random variables $T_{i,\,p}$, $T'_{i,\,p}$, and $T_{ij,\,p}$, only by $O(\omega^5\sqrt{n})$ and $O(\omega^4)$, respectively, while the expectation squared is of the order of $\omega^6 n$.

We will give the proof of statement (iii) for $* = +$ only. The proof remains the same for $* = +'$ and $* = +''$, while for $* = -$, in the subsequent inequalities (9) and (10) we have an extra factor of 2 in front of the RHS and an extra factor of 1/2 in the exponent. This, however, has no effect on the proof.

By Lemma 1(iii)(b) we know that for all $\omega \leqslant v \leqslant n$ and all $1 \leqslant i \neq j \leqslant k$,

$$x^+(v, i, j) \sim \frac{v}{k^2},$$

while

$$x^+(v, i, i) \sim \frac{v}{2k^2}.$$

Thus, with some room to spare, for all v and all i, j, $x^+(v, i, j) < 2n/k^2$. Consequently, $X^+_{pp}(v, i, j)$ is bounded from above by the binomially distributed random variable B with expectation $\mu = 2np^2/k^2$.

Set $Y_s = |\{v\colon X^+_{pp}(v, i, j) > s\mu\}|$, $s = 2, 3, \ldots$. By Chernoff's inequality,

$$\mathbb{E}Y_2 = n\mathbb{P}(X^+_{pp}(v, i, j) > 2\mu) \leqslant n\,\mathbb{P}(B > 2\mu) < ne^{-\mu/3} \tag{9}$$

and, more easily, for $s \geqslant 3$,

$$\mathbb{E}Y_s = n\,\mathbb{P}(X^+_{pp}(v, i, j) > s\mu) \leqslant n\,\mathbb{P}(B > s\mu) < n\binom{(2n)/k^2}{s\mu} p^{2s\mu} < n(e/s)^{s\mu}. \tag{10}$$

Hence, by Markov's inequality, we have $\mathbb{P}(Y_2 > ne^{-\mu/6}) = o(1)$ and, for $s \geqslant 3$,

$$\mathbb{P}(Y_s > n(e/s)^{s\mu/2}) < (e/s)^{s\mu/2} = p_s.$$

Observe that $\sum_s p_s = o(1)$ and hence, with probability tending to 1,

$$\sum_{v \notin D^+_{pp}} X^+_{pp}(v, i, j) < \sum_{s=2}^{\infty} Y_s(s+1)\,\mu$$

$$< ne^{-\mu/6}(3\mu) + \sum_{s=3}^{\infty} n(e/s)^{s\mu/2}\,(s+1)\,\mu = o(n).$$

The proof of (b) follows the same lines as the proof of part (a) and is based on the observation that

$$\mathbb{P}(X^+_{ppp}(v, i, j) > s\mu) = p\mathbb{P}(X^+_{pp}(v, i, j) > s\mu).$$

This follows immediately from the fact that the random variable $X^+_{pp}(v, i, j)$ is independent of the event "$v \in [n]_p$." ▮

Let $\mathscr{E}$ denote the event that $[n]_p$ admits a 2-coloring $[n]_p = R \cup B$ which has fewer than $\alpha n^2 p^3$ monochromatic Schur triples.

In what follows we will introduce several constants that need to satisfy certain inequalities. To make sure that all the requirements can be fulfilled at the same time, we now give a particular feasible choice of these constants, with no attempt to make them optimal.

Set $\alpha = (0.99)\,2^{-59}3^{-12}10^{-6}$, $k = 2^{18}3^4 10^2$, $\delta = 2^{-7}$ and $\gamma = 0.01$. Let us call a 2-coloring $[n]_p = R \cup B$ *unbalanced* if for some i we have either

$$|(N'_i)_p \cap R| \leqslant \frac{\delta np}{k} \qquad \text{or} \qquad |(N'_i)_p \cap B| \leqslant \frac{\delta np}{k}. \tag{11}$$

For each $i = 1, ..., k$, denote by $\mathscr{E}_i \subset \mathscr{E}$ the event that $[n]_p$ has an unbalanced 2-coloring (on $(N'_i)_p$) with fewer than $\alpha n^2 p^3$ monochromatic Schur triples.

LEMMA 3. *For each $i = 1, ..., k$, $Pr[\mathscr{E}_i] = o(1)$ as $n \to \infty$.*

Proof. Let $[n]_p = R \cup B$ be a 2-coloring which is unbalanced on $(N'_i)_p$. We can assume without loss of generality that $|(N'_i)_p \cap R| \leqslant (\delta np)/k$. By Lemma 2(iii)(b) there are, with probability $1 - o(1)$, at most

$$2\left(\frac{\delta np}{k}\,2\mu + o(np)\right) \sim 4\,\frac{\delta n^2 p^3}{k^3}$$

Schur triples in $(N_i')_p$ involving a red element. (The factor of 2 comes here from the fact that a Schur triple involving v may be of the form $x+y=v$ or $x-y=v$.) However, this time by Lemma 2(ii), there are, with probability $1-o(1)$,

$$T'_{i,p} \sim \frac{n^2p^3}{16k^3} \text{ Schur triples in } (N_i')_p.$$

Thus, at least

$$(1+o(1))\left(\frac{1}{16}-4\delta\right)\frac{n^2p^3}{k^3} \geqslant \alpha n^2p^3$$

of them are blue, which is what is needed. (Note, that the factor 0.99 in the definition of α, accommodates the inaccuracy caused by the term $o(1)$.) This proves Lemma 3. ▌

Now, it remains to show that the event $\mathscr{E}' = \mathscr{E} \setminus \bigcup_{i=1}^{k} \mathscr{E}_i$ saying that $[n]_p$ has a *balanced* (i.e., not unbalanced) 2-coloring $[n]_p = R \cup B$ with fewer than αn^2p^3 monochromatic Schur triples is unlikely. For $B' \subset B$, $R' \subset R$ we define:

$$e(B', R)^+ = \sum_{\pi=(x,y) \in B' \times R'} I_\pi^+ \qquad \text{where} \quad I_{(x,y)}^+ = \begin{cases} 1 & \text{if } \; x+y \in [n]_p, \\ 0 & \text{otherwise} \end{cases}$$

and

$$e(B', R')^- = \sum_{\pi=(x,y) \in B' \times R'} I_\pi^-$$

where

$$I_{(x,y)}^- = \begin{cases} 1 & \text{if } \; |x-y| \in [n]_p, \ |x-y| \neq \min(x,y) \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$R^{(1)} = R \cap \left[\frac{n}{2}\right], \qquad R^{(2)} = R - R^{(1)}$$

$$B^{(1)} = B \cap \left[\frac{n}{2}\right], \qquad B^{(2)} = B - B^{(1)}.$$

Main Lemma. *For every balanced 2-coloring $[n]_p = B \cup R$ the following inequalities hold with probability $1-o(1)$:*

(i) $e(B, R)^- < (1+\gamma)\,|B|\,|R|\,p$

(ii) $e(B^{(1)}, R^{(1)})^+ < (1+\gamma)\,|B^{(1)}|\,|R^{(1)}|\,p$

(iii) $e(B^{(1)}, R^{(2)})^+ + e(B^{(2)}, R^{(1)})^+ < (1+\gamma)(|B^{(1)}|\,|R^{(2)}| + |B^{(2)}|\,|R^{(1)}|)\,p$

Note that (ii) and (iii) imply

(iv) $e(B, R)^+ < (1+\gamma)\,|B|\,|R|\,p.$

Note also that this lemma is in fact true not only for $\gamma = 0.01$ but for any $\gamma > 0$ provided k is sufficiently large.

Before proving the Main Lemma, we first show why this implies Theorem 1, i.e., why it implies the event $\mathscr{E}'$. The proof will mimic our Goodman-type argument presented for the deterministic case in Section 2.

Proof of Theorem 1. First, we note that the number T_p of Schur triples in $[n]_p$ satisfies (directly by Chebyshev's inequality, or by Lemma 2(ii)) $T_p \sim (1/4)\,n^2p^3$ with probability $1 - o(1)$. Let A and M denote the numbers of achromatic and monochromatic Schur triples in $[n]_p$ with respect to a fixed balanced coloring $[n]_p = R \cup B$, and let $r = |R|, b = |B|$. Thus, $M = T - A$.

Let us count the number of pairs (π, t) where $\pi \in B \times R$ and t is a Schur triple in $[n]_p$ containing (the entries of) π. There are clearly $2A$ such pairs. On the other hand, their number is also equal to $e(B, R)^+ + e(B, R)^-$ so that

$$A = (1/2)(e(B, R)^+ + e(B, R)^-).$$

Recall that $\gamma = 0.01$, set $\beta = 0.1$ and distinguish two cases:

Case I. $br \leqslant (1/4 - \beta^2)\,n^2p^2$, or equivalently, say, $r \geqslant (1/2 + \beta)\,np$. Then

$$\begin{aligned} M = T - A &= T - \frac{1}{2}(e(B, R)^+ + e(B, R)^-) \\ &\geqslant T - (1+\gamma)\,brp \quad \text{(by (i) and (iv))} \\ &\geqslant (1+o(1))\,\frac{n^2p^3}{4} - (1+\gamma)\left(\frac{1}{4} - \beta^2\right) n^2p^3 \\ &\geqslant (1+o(1))\left(\frac{1}{4} - (1+\gamma)\left(\frac{1}{4} - \beta^2\right)\right) n^2p^3 \\ &= (1+o(1))(\beta^2(1+\gamma) - \gamma/4)\,n^2p^3 \\ &\geqslant (1+o(1))(\beta^2 - \gamma/4)\,n^2p^3 \\ &\geqslant (1+o(1))\,\alpha n^2p^3. \end{aligned} \tag{12}$$

Case II.

$$(\tfrac{1}{2}-\beta)\, np \leqslant b \leqslant r \leqslant (\tfrac{1}{2}+\beta)\, np.$$

Since $e(B^{(2)}, R^{(2)})^+ = 0$,

$$\begin{aligned}2A &= e(B, R)^- + e(B^{(1)}, R^{(1)})^+ + e(B^{(1)}, R^{(2)})^+ + e(B^{(2)}, R^{(1)})^+ \\ &\leqslant (1+\gamma)(br + b_1 r_1 + b_1 r_2 + b_2 r_1)\, p \\ &= (1+\gamma)(2br - b_2 r_2)\, p. \qquad (13)\end{aligned}$$

Subcase 1.

$$b_2 = \lambda np, \qquad \tfrac{1}{20} \leqslant \lambda \leqslant \tfrac{9}{20}.$$

Thus,

$$b_2 r_2 \geqslant (1+o(1))\, \lambda(1/2-\lambda)\, n^2p^2 \geqslant (1+o(1))\, \tfrac{9}{400} n^2 p^2$$

so that by (13)

$$\begin{aligned}M = T - A &= (1+o(1))\, \frac{n^2p^3}{4} - \frac{(1+\gamma)}{2}\,(2br - b_2 r_2)\, p \\ &\geqslant (1+o(1)) \left(\frac{1}{4} - (1+\gamma)\left(\frac{1}{4} - \frac{9}{800}\right)\right) n^2 p^3 \\ &= (1+o(1)) \left(\frac{9-191\gamma}{800}\right) n^2 p^3 > \alpha n^2 p^3\end{aligned}$$

as desired.

Subcase 2.

$$b_2 = \lambda np, \quad \lambda < 1/20 \quad (\text{the case } \lambda > 9/20 \text{ is symmetrical}).$$

Thus,

$$\begin{aligned}b_1 = b - b_2 &\geqslant \left(\frac{1}{2} - \beta\right) np - \frac{1}{20}\, np \\ &= \left(\frac{9}{20} - \beta\right) np = \left(\frac{1}{2} + \left(\frac{4}{10} - 2\beta\right)\right) \frac{np}{2},\end{aligned}$$

and we may follow the argument of case I applied to $[n/2]_p$ with β replaced by $(4/10)-2\beta$. Hence, we have

$$\begin{aligned} M &\geqslant (1+o(1))\,\tfrac{1}{4}((\tfrac{4}{10}-2\beta)^2-\gamma/4)\,n^2p^3 \\ &\geqslant \alpha n^2p^3. \quad \blacksquare \end{aligned}$$

Proof of Main Lemma. We will prove (i) in detail and then only discuss the proofs of (ii) and (iii).

For $\pi=(x, y)\in B\times R$, let

$$\tilde{I}^-_{(x,\,y)}=\begin{cases} 1 & \text{if} \quad |x-y|\in[n]_p \quad \text{and} \\ & x,\, y,\, |x-y| \text{ belong to 3 different classes,} \\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\sum_{\pi=(x,\,y)\in B\times R}(I^-_\pi-\tilde{I}^-_\pi)\leqslant 2T_{0,\,p}$$

where $T_{0,\,p}=\sum_i T_{i,\,p}+\sum_{i\neq j}T_{i,\,j,\,p}$ is the number of Schur triples in $[n]_p$ with at least two elements in the same class. By Lemma 2(ii), with probability close to 1, $T_{0,\,p}<(1/k)\,n^2p^3$, and so,

$$\sum_{\pi=(x,\,y)\in B\times R}(I^-_\pi-\tilde{I}^-_\pi)\geqslant\tfrac{1}{2}\gamma brp$$

since, with some room to spare, $br>\delta^2n^2p^2$ and $\gamma>4/\delta^2k$.

Set

$$\tilde{e}(B, R)^-=\sum_{\pi=(x,\,y)\in B\times R}\tilde{I}^-_\pi.$$

Then $0\leqslant e(B, R)^- - \tilde{e}(B, R)^-\leqslant(1/2)\,\gamma brp$.

Suppose, to the contrary, that there exists a balanced coloring such that $e(B, R)^->(1+\gamma)\,brp$. Then, also $\tilde{e}(B, R)^->(1+1/2\gamma)\,brp$.

Now, for $i\neq j$, set

$$\tilde{e}(B_i, R_j)^-=\sum_{\pi=(x,\,y)\in B\times R}\tilde{I}^-_\pi.$$

Consequently,

$$\sum_{i\neq j}\tilde{e}(B_i, R_j)^-=\tilde{e}(B, R)^->(1+\tfrac{1}{2}\gamma)\,brp>(1+\tfrac{1}{2}\gamma)\sum_{i\neq j}b_ir_jp.$$

and, trivially, there exist $i \neq j$ such that

$$\tilde{e}(B_i, R_j)^- > (1 + \tfrac{1}{2}\gamma)\, b_i r_j p. \tag{14}$$

Set further

$$\hat{I}_{(x,y)} = \begin{cases} \tilde{I}_{(x,y)} & \text{if } \ |x - y| \in D^-_{ppp}(i, j) \cap D^-_{ppp}(j, i), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{e}(B_i, R_j)^- = \sum_{\pi = (x, y) \in B \times R} \hat{I}^-_\pi,$$

where the sets $D^-_{ppp}(i, j)$ were defined in Lemma 2(iii). By our assumptions on p and on the coloring, we have that

$$np = o(n^2 p^3) = o(b_i r_j p).$$

Then by Lemma 2(iii)(b), with probability $1 - o(1)$,

$$\begin{aligned} \tilde{e}(B_i, R_j)^- - \hat{e}(B_i, R_j)^- &\leqslant \sum_{v \notin D^-_{ppp}(i, j)} X^-_{ppp}(v, i, j) + \sum_{v \notin D^-_{ppp}(j, i)} X^-_{ppp}(v, j, i) \\ &= o(np) = o(b_i r_j p) \end{aligned}$$

and, hence, by (14) we infer that,

$$\hat{e}(B_i, R_j)^- = \tilde{e}(B_i, R_j)^- - o(b_i r_j p) > (1 + \tfrac{1}{3}\gamma)\, b_i r_j p. \tag{15}$$

We will show that the event $\mathscr{A}$, that there is a balanced coloring satisfying the above inequality for some i, j, is unlikely.

We fix $i \neq j$ and apply a variant of the two-round exposure technique. In round 1 we generate random subsets $(N_i)_p$ and $(N_j)_p$ only.

Let $\mathscr{B}$ be the event that, after round 1, Lemma 2 holds for these i, j. By that lemma, we already know that $\mathbb{P}(\mathscr{B}) \to 1$ as $n \to \infty$.

Given a set $F \subseteq N_i \cup N_j$, $F \in \mathscr{B}$, we define, for every coloring $F = B' \cup R'$, $B' = B_i \cup B_j$ and $R' = R_i \cup R_j$, the event $\mathscr{A}_{(B', R')}$ that $\mathscr{A}$ holds conditioned on $(N_i)_p \cup (N_j)_p = F$ and with the coloring (B', R') on F. Then, routinely,

$$\mathbb{P}(\mathscr{A}) = \sum_{F \in \mathscr{B}} \mathbb{P}(\mathscr{A} \mid (N_i)_p \cup (N_j)_p = F)\, \mathbb{P}((N_i)_p \cup (N_j)_p = F) + o(1)$$

and

$$\mathbb{P}(\mathscr{A} \mid (N_i)_p \cup (N_j)_p = F) \leqslant \sum_{(B',\, R')} \mathbb{P}(\mathscr{A}_{(B',\, R')} \mid (N_i)_p \cup (N_j)_p = F)$$
$$\leqslant 2^{|F|} \mathbb{P}(\mathscr{A}_{(B'_0,\, R'_0)} \mid (n_i)_p \cup (N_j)_p = F)$$

where (B'_0, R'_0) maximizes the above conditional probability.

Since $F \in \mathscr{B}$, we have, by Lemma 2(i) that, again with an extra cushion, $|F| < 3np/k$. Thus, all we need to show is that for any $F \in \mathscr{B}$ and any partition $F = B' \cup R'$,

$$\mathbb{P}(\mathscr{A}_{(B',\, R')} \mid (N_i)_p \cup (N_j)_p = F) \leqslant e^{-3np/k},$$

say.

For every $v \in [n]$, let f_v be the number of pairs $\{u, w\}$ such that $u \in B_i$, $w \in R_j$ and $v = |u - w|$. Let us set $D = \{v \in [n] \backslash (N_i \cup N_j) : f_v < 4\mu\}$. Notice that $f_v \leqslant X^-_{pp}(v, i, j) + X^-_{pp}(v, j, i)$.

Entering the second round of exposure, and focusing on the set $D_p = \{v \in D : v \in [n]_p\}$, let

$$J(v) = \begin{cases} 1 & \text{if} \quad v \in D_p, \\ 0 & \text{otherwise,} \end{cases}$$

and let $Z = \sum_{v \in D} f_v J(v)$. Since $D^-_{pp}(i, j) \cap D^-_{pp}(j, i) \subseteq D$, then we have $Z \geqslant \hat{e}(B_i, R_j)^-$. Moreover,

$$\mathbb{E}(Z) = p \sum_{v \in D} f_v \leqslant p \sum_{v \in [n]} f_v = p b_i r_j.$$

Hence, the event $\mathscr{A}_{(B',\, R')}$, saying that inequality (15) holds with $(N_i)_p \cup (N_j)_p = B' \cup R'$, implies that

$$Z \geqslant \left(1 + \frac{\gamma}{3}\right) \mathbb{E}(Z).$$

To prove that this inequality is very unlikely, we notice that Z is a sum of independent, nonnegative random variables, all smaller than 4μ, and apply the following version of Hoeffding's inequality (cf. [6], Cor. (5.2)(b), p. 162) to $Z/4\mu$:

Lemma 4. *Let* $X_1, ..., X_n$ *be independent random variables satisfying* $0 \leqslant X_l \leqslant 1$ *for each* $l = 1, ..., n$ *and let* $v = \mathbb{E}(\sum_{l=1}^n X_l)$. *Then*

$$\mathbb{P}\left(\sum_{l=1}^{n} X_l \geqslant (1 + \varepsilon)\, v\right) \leqslant \exp\{-\tfrac{1}{3}\varepsilon^2 v\}$$

and

$$\mathbb{P}\left(\sum_{l=1}^{n} X_l \leqslant (1-\varepsilon)\, v\right) \leqslant \exp\{-\tfrac{1}{2}\varepsilon^2 v\}.$$

The first inequality of Lemma 4 implies that

$$\mathbb{P}\left(Z \geqslant \left(1+\frac{\gamma}{3}\right) \mathbb{E}(Z)\right) < \exp\left\{-\frac{\gamma^2}{27}\, \mathbb{E}\left(\frac{Z}{4\mu}\right)\right\}. \tag{16}$$

It remains to bound $\mathbb{E}(Z)$ from below. We have, by Lemma 2(iii)(a),

$$\sum_{v \in [n] \setminus D} f_v \leqslant \sum_{v \notin D^-_{pp}(i,\, j)} X^-_{pp}(v, i, j) + \sum_{v \notin D^-_{pp}(j,\, i)} X^-_{pp}(v, j, i) = o(n)$$

and, by the definition of the set D and the upper bound on the size of set $F = (N_i)_p \cup (N_j)_p$,

$$\sum_{v \in F \cap D} f_v < \frac{3n}{k}\,(4\mu).$$

Hence, recalling that $\mu = (2n/k^2)\, p^2$ and $np = o(pb_i r_j)$,

$$E(Z) > pb_i r_j - \frac{3n}{k}\,(4\mu)\, p - o(np) \geqslant \frac{1}{2}\, pb_i r_j \tag{17}$$

since $k\delta^2 \geqslant 50$.

Finally, by (16) and (17),

$$\mathbb{P}\left(Z \geqslant \left(1+\frac{\gamma}{3}\right) \mathbb{E}(Z)\right) < \exp\left\{-\frac{\gamma^2}{27}\, pb_i r_j / 8\mu\right\} \leqslant e^{-3np/k}$$

since $\gamma^2\delta^2 k > 2^4 3^4$. This proves (i).

The proof of (ii) follows the same lines, but everything is scaled down to the lower half of $[n]$. There is a slight complication in carrying out the proof of (iii) along the same argument. The problem is that the balanced coloring hypothesis does not apply to the upper halves of the classes N_i's and, therefore, we are not in position to prove inequalities like $e(B^{(1)}, R^{(2)})^+ < (1+\gamma)\, |B^{(1)}|\, |R^{(2)}|\, p$, since we cannot guarantee that $b_i^{(1)} r_j^{(2)} = \Omega(n^2 p^2)$. Nevertheless, for any $i \neq j$, we have, for a balanced coloring, that $b_i^{(1)} r_j^{(2)} + b_i^{(2)} r_j^{(1)} > \delta (np)^2/4$, since $r_j^{(2)} + b_i^{(2)} \sim np/2k$ and $b_i^{(1)}, r_j^{(1)} > \delta np/2k$, and the proof goes through. This completes our discussion of the proof of Theorem 1. ∎

4. PROOF OF THEOREM 2

For the proof of Theorem 2 we assume that $p = cn^{-1/2}$, where c is a sufficiently small constant. Our proof will consist of two statements, one deterministic, saying that for any set $F \subseteq [n]$, the Schur property, i.e., the property that every 2-coloring of F contains a monochromatic Schur triple, implies the existence of a certain structure in F, while the probabilistic statement will almost surely exclude that structure from the random set $[n]_p$. We shall first need a few hypergraph definitions. All hypergraphs will be 3-uniform, i.e., all their edges have size 3. A *simple path* is a hypergraph consisting of edges $E_1, ..., E_l$, $l \geqslant 1$, such that

$$|E_i \cap E_j| = \begin{cases} 1 & \text{if} \quad j = i+1,\ i = 1, ..., l-1 \\ 0 & \text{otherwise.} \end{cases}$$

A *fairly simple cycle* is a hypergraph which consists of a simple path $(E_1, ..., E_l)$, $l \geqslant 2$, and an edge E_0 such that

$$|E_0 \cap E_i| = \begin{cases} 1 & \text{if} \quad i = 1 \\ 0 & \text{for} \quad i = 2, ..., l-1 \\ s & \text{if} \quad i = l, \end{cases}$$

where $1 \leqslant s \leqslant 2$. If $s = 1$ then we call it *simple*. A fairly simple cycle which is not simple cycle, i.e. $s = 2$, will be called *spoiled*.

A simple path P of a hypergraph H is called *spoiled* if it is not an induced subhypergraph of H, i.e., there is an edge E in H such that $E \notin E(P)$ but $E \subset V(P)$.

A subhypergraph H_0 of H is said to have a *handle* if there is an edge E in H such that $|E \cap V(H_0)| = 2$.

For a set of integers F, let $H(F)$ be the hypergraph with the vertex set F whose edges are the Schur triples in F.

Deterministic Lemma. *If F has Schur property then the hypergraph $H(F)$ contains either a fairly simple cycle with a handle or a spoiled simple path.*

Probabilistic Lemma. *If $p = cn^{-1/2}$ then, with probability tending to* 1 *as $n \to \infty$, the random hypergraph $H([n]_p)$ contains neither a fairly simple cycle with a handle nor a spoiled simple path.*

Proof of the Deterministic Lemma. Assume that F has the Schur property. This is equivalent to saying that the chromatic number of $H(F)$ is at least 3. We may assume that $H(F)$ is edge-critical with respect to that property since otherwise we could replace $H(F)$ with its 3-edge-critical subgraph, ignoring some Schur triples in F. As such, it satisfies the following property.

Fact. If H is a 3-edge-critical hypergraph then for every edge $E \in H$ and for every vertex $v \in E$ there is $E' \in H$ such that $E \cap E' = \{x\}$.

Let P be the longest simple path in $H = H(F)$. By the Fact, P contains at least two edges of H. Let x and y be two vertices which belong to only the first edge of P, and let E_x and E_y be two edges of H (read: Schur triples) whose existence is guaranteed by the Fact, i.e., $E_z \cap E_1 = \{z\}$, $z = x, y$.

By the maximality of P, $h_z = |V(P) \cap E_z| \geqslant 2$, $z = x, y$. Let $i_z = \min\{i \geqslant 2: E_z \cap E_i \neq \emptyset\}$, $z = x, y$, and assume that, say, $i_y \leqslant i_x$. If $h_z = 3$ for some z, then P is a spoiled simple path. Otherwise, the edges $E_1, ..., E_{i_x}, E_x$ form a fairly simple cycle for which E_y is a handle. ∎

The Proof of Probabilistic Lemma. Let X, Y, Z, and W be random variables counting, respectively, simple paths of length at least $B \log n$, spoiled cycles, simple cycles of length less than $B \log n + 1$ with handles, and spoiled simple paths of length less than $B \log n$, in the random hypergraph $H([n]_p)$, where $B = B(c)$ is a sufficiently large constant. Straightforward estimates show that their expectations all converge to 0 as $n \to \infty$. Indeed,

$$\mathbb{E}(X) = O\left(\sum_{t > B \log n} n^2 n^{t-1} p^{3+2(t-1)}\right) = O\left(np \sum_{t > B \log n} c^t\right) = o(1).$$

To estimate $\mathbb{E}(Y)$, we begin with a pair of edges which spoil some cycle, i.e., which intersect each other in 2 elements, and continue along the cycle until the last edge closes it by sharing one vertex with both the previous and the first edge. Thus,

$$\mathbb{E}(Y) < O\left(\sum_{t > 2} n^2 p^4 (np^2)^{t-3}\, p\right) = o(1).$$

Similarly,

$$\mathbb{E}(Z) = O\left(\sum_{t=3}^{B \log n} n^t p^{2t} (\log n)^2\, p\right) = o(1),$$

where the logarithmic factor represents the number of choices of the two elements at which a handle is attached to the cycle.

The spoiled simple paths can be classified into two types: those with at least one spoiling edge intersecting an edge of the path in two vertices, and the others. Let us denote their numbers by W_2 and W_1, respectively. We have $W = W_1 + W_2$ and, clearly, $W_2 > 0$ implies $Y > 0$. Thus we need to worry only about W_1. However, if a spoiling edge E intersects each edge of a simple path P in at most one vertex, then there is a subhypergraph consisting of a simple cycle C_1 (made by E and a segment of P between two consecutive intersections with E) and a simple path P_1 with its end-edges intersecting two consecutive edges of C_1 each in one vertex, but otherwise being disjoint from C_1. Let U count such configurations in $H([n]_p)$. Thus, $W_1 > 0$ implies that $U > 0$ and we need to estimate $\mathbb{E}(U)$. Letting t_1 represent the number of edges in C_1, and t_2 the number of edges in P_1, we have,

$$\mathbb{E}(U) = O\left(\sum_{t_1 \geqslant 3} \sum_{t_2 \geqslant 1} c^{t_1}(np^2)^{t_2 - 1}\, p\right) = O(p) = o(1).$$

Hence, by Markov's inequality, $\mathbb{P}(X = Y = Z = W = 0) \to 1$ as $n \to \infty$, which was to be proved. ∎

5. CONCLUDING REMARKS

We strongly believe that Theorem 1, like Theorem 3, should hold even if $\omega(n)$ is replaced by a sufficiently large constant C. However, to improve Theorem 1, we would need another approach. If one would like to follow the approach designed for arithmetic triples, one should first come up with some sort of structural condition guaranteeing that a set $D \subseteq [n]$ satisfying it contains many Schur triples. On the other hand, one should be able to show that the set D of elements extending monochromatic pairs of elements of colored $[n]_p$ to Schur triples satisfies this condition with probability tending to 1 as $n \to \infty$. At this moment we do not see any reasonable candidate for such a condition.

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