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TITLE— Some Remarks on Certain Sums of Rational Functions

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ABSTRACT

In a recent note concerned with problems arising in nonnumerical computer methods, S. C. Johnson asked for which rational functions $r(x,y)$ is the function $\sum_{k=1}^{n} r(n,k)$ also a rational function of n. In this memorandum we demonstrate a number of techniques which can be used to establish the nonrationality of the above sum for a variety of functions $r(x,y)$.

10 pages of text
2 references

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MEMORANDUM FOR FILE

INTRODUCTION

In a recent note concerned with problems arising in nonnumerical computer methods, S. C. Johnson [2] raised the following interesting question:

Query: For which rational functions* $r(x,y)$ is the function

$$R(n) = \sum_{k=1}^{n} r(n,k)$$

a rational function of n?

In particular, he proposed the following

Conjecture: $R(n)$ is a rational function of n if and only if $r(x,y)$ can be written as

$$r(x,y) = s(x,y+1) - s(x,y)$$

for some rational function $s(x,y)$.

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*Here, all rational functions will be considered to be over the field of real numbers.

No counterexamples to this conjecture are known at present. Johnson points out that the conjecture, if true, would have the following corollaries:

(i) $\sum_{k=1}^{n} \frac{1}{k+p(n)}$ is not a rational function of n;

(ii) $\sum_{k=1}^{n} \frac{1}{k^2+p^2(n)}$ is not a rational function of n.

It is our purpose in this memorandum to illustrate several approaches which succeed in establishing results similar to (i) and (ii), thus providing evidence in support of the conjecture. The methods presented will not be fully exploited but rather they will be applied to special cases from which it is left to the reader to decide under what generality the same ideas carry through.

A PRELIMINARY REMARK

The following result will be useful.

Proposition 1: Suppose $r(x,y)$ has rational coefficients and suppose

$$\sum_{k=1}^{n} r(n,k) = P(n)/Q(n)$$

is a rational function of n where P and Q are fixed polynomials and the leading coefficient q_0 of Q is 1. Then all the coefficients of P and Q are rational.

Proof: Let s denote deg P + deg Q. Consider the s + 1 equations

$$Q(m) \sum_{k=1}^{m} r(m,k) = P(m), \qquad 1 \leq m \leq s + 1.$$

This is a system of s + 1 linear equations in the s + 1 coefficients of the polynomials P and Q (recall that q_0 is 1). Moreover, the coefficients of this system, the r(m,k), are rational. Since a solution to this system of equations exists by hypothesis, then there is a suitable maximal independent subset of the system whose solution is given by Cramer's rule, i.e., a ratio of two determinants with rational entries (again, since $q_0 = 1$ and the r(m,k) are rational). Hence, all the coefficients of P and Q are rational. In fact, we can assume they are integral.

In essentially the same way, we can establish

Proposition 1': Suppose r(x,y) is a rational function with coefficients in some extension field $\mathbb{Q}(\alpha)$ of the rational numbers $\mathbb{Q}$. Further, suppose

$$\sum_{k=1}^{n} r(n,k) = P(n)/Q(n)$$

is a rational function of n where P and Q are fixed polynomials and the leading coefficient of Q is 1. Then all the coefficients of P and Q belong to $\mathbb{Q}(\alpha)$.

SOME SPECIAL SUMS

<u>Proposition 2</u>: $\sum_{k=1}^{n} \frac{1}{n+k}$ is not a rational function of n.

<u>Proof</u>: We give two proofs of this.

I. Consider the sum $\sum_{k=1}^{n} \frac{1}{n+k}$ written as a (reduced) rational number p_n/q_n. It is easily checked that in any finite sum of unit fractions $p^*/q^* = \sum_{k=1}^{n} \frac{1}{m_k}$, with m_k positive integers, if for some prime p and positive integer r, p^r divides <u>exactly one</u> of the m_k then p^r must divide q^*. In particular, therefore, any prime p with $n + 1 \leq p \leq 2n$, must divide q^*. But by the prime number theorem, for any $\varepsilon > 0$ there are asymptotically

$$\frac{2n}{\log 2n} - \frac{n}{\log n} \geq \frac{2n}{(1+\varepsilon)\log n} - \frac{n}{\log n} = \frac{n}{\log n}\left(\frac{2}{1+\varepsilon} - 1\right)$$

primes between $n + 1$ and $2n$ for n sufficiently large. Since all these primes are $> n$ then this product is $> n^{\frac{n}{\log n}\left(\frac{2}{1+\varepsilon} - 1\right)}$. But this number must divide q_n and since it grows faster than any power of n then $\sum_{k=1}^{n} \frac{1}{n+k}$ cannot be the ratio of two polynomials in n.

II. Assume $\sum_{k=1}^{n} \frac{1}{k+n} = \frac{P(n)}{Q(n)}$, with P,Q fixed polynomials in n. By Proposition 1, we can assume P and Q have integral coefficients. An elementary estimate shows that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k+n} = \log 2.$$

But this must also equal $\lim_{n\to\infty} P(n)/Q(n)$. However, this implies deg P = deg Q and the ratios of the leading coefficients of P and Q must be log 2, which is impossible if P and Q have integral coefficients (by the well-known irrationality of log 2).

By using Proposition 1′ in the argument of II and the transcendence of log 2, we can also establish the following result.

Proposition 2′: If α is algebraic over $\mathbb{Q}$ then $\sum_{k=1}^{n} \frac{1}{n+k+\alpha}$ is not a rational function of n.

Proposition 3: $\sum_{k=1}^{n} \frac{1}{n^r+k}$ is not a rational function of n for any integer $r \geq 1$.

Proof: Again we look at the powers of the primes which divide the n consecutive integers $n^r + k$, $1 \leq k \leq n$. For any prime p, if $p^a > n$ then p^a can divide at most one of the n denominators and hence, by a previous remark, p^a must also divide the denominator q_n of the sum $p_n/q_n = \sum_{k=1}^{n} \frac{1}{n^r+k}$. Hence, we restrict our attention to those numbers $n^r + k$ which are divisible at most $p^b \leq n$. An easy counting argument shows that the sum of all the exponents of such p is no more than $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$

$\leq \frac{n}{p-1}$. The product of all these is at most $\prod_{p\leq n} p^{\frac{n}{p-1}} = P_n$

and consequently, since $\prod_{k=1}^{n} (n^r+k) \geq n^{rn}$ then we must have

$$q_n \geq n^{rn}/P_n.$$

However, we have the following estimate [1] for P_n:

$$P_n \leq n^{(1+\varepsilon)n}$$

for any $\varepsilon > 0$ provided n is sufficiently large. Hence

$$q_n \geq n^{(r-1-\varepsilon)n}$$

which clearly grows too fast to be generated by a polynomial.

By modifying the preceding techniques slightly we can establish

Proposition 4: $\sum_{k=1}^{n} \frac{1}{f(n)+k}$ is not a rational function of n for any polynomial f(n) with rational coefficients.

The new problem which comes in the proof here is that after clearing the fractions in the denominator $\frac{1}{f(n)+k}$ we are left with a sum of the form $\sum_{k=1}^{n} \frac{1}{g(n)+ck}$ where g has integral coefficients and c is an integer. The preceding estimates on the total power of a prime p which we may lose when summing is now changed only for those p which divide c. Since there are just finitely many of these then for n sufficiently large the estimates will not be significantly affected.

<u>Proposition 5</u>: $\sum_{k=1}^{n} \frac{1}{n^2+k^2}$ is not a rational function of n.

<u>Proof</u>: Suppose it is, say it equals $P(n)/Q(n)$, all with integral coefficients. Thus we have

$$\sum_{k=1}^{n} \frac{n}{n^2+k^2} = \frac{nP(n)}{Q(n)} .$$

But

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\frac{1}{n}}{1+\left(\frac{k}{n}\right)^2} = \int_0^1 \frac{dx}{1+x^2} = \text{arc tan } 1 = \frac{\pi}{4} .$$

On the other hand this limit is also equal to $\lim_{n\to\infty} \frac{nP(n)}{Q(n)}$, which, because it exists and is not 0, $\pm\infty$, must be the ratio of the leading coefficients of P and Q. This contradicts the well-known irrationality of π.

In general, the technique of replacing the limit of the sum as n tends to infinity by an appropriate integral would appear to be a powerful technique. We have not pursued this direction any further, however.

Another way in which these sums may differ from rational functions is illustrated in the next result.

Proposition 6: $\sum_{k=1}^{n} \frac{1}{n^2+k^3}$ is not a rational function of n.

Proof: Consider the sum $\sum_{k=1}^{n} \frac{n}{n^2+k^3}$. We have

$$\sum_{k=1}^{n} \frac{n}{n^2+k^3} = \sum_{k<\frac{n}{\log n}} \frac{n}{n^2+k^3} + \sum_{\frac{n}{\log n}\leq k\leq n} \frac{n}{n^2+k^3}$$

$$\leq \sum_{k<\frac{n}{\log n}} \frac{1}{n} + n \frac{n}{n^2+\left(\frac{n}{\log n}\right)^3}$$

$$\leq \frac{1}{\log n} + \frac{1}{1+\frac{n}{\log^3 n}} \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand,

$$\sum_{k=1}^{n} \frac{n^2}{n^2+k^3} \geq \sum_{k\leq n^{2/3}} \frac{n^2}{n^2+k^3} \geq \sum_{k\leq n^{2/3}} \frac{n^2}{n^2+n^2} \geq \frac{1}{2}\, n^{2/3} \to \infty \text{ as } n \to \infty.$$

Therefore, $\sum_{k=1}^{n} \frac{1}{n^2+k^3}$ cannot be a rational function of n because it does not behave asymptotically like an integral power of n. This approach can be used in a variety of cases in which r(n,1) and r(n,n) have different orders of growth.

We conclude with an example having a different flavor.

<u>Proposition 7</u>: $\sum_{k=1}^{n} \frac{1}{n+k+\alpha}$ is not a rational function for any transcendental number α.

<u>Proof</u>: Assume

$$\sum_{k=1}^{n} \frac{1}{n+k+\alpha} = \frac{P(n)}{Q(n)}$$

where P and Q are polynomials in n. We can apply an argument similar to the one used in Proposition 1 to show that the coefficients in P and Q can be taken to be <u>polynomials</u> in α with integral coefficients. Hence we have $P(n) = P^*(n,\alpha)$, $Q(n) = Q^*(n,\alpha)$ where P^* and Q^* are polynomials with integral coefficients. Now,

$$\sum_{k=1}^{n} \frac{1}{n+k+\alpha} = \sum_{k=1}^{n} \prod_{j\neq k} (n+j+\alpha) \Big/ \prod_{k=1}^{n} (n+k+\alpha) \equiv \frac{U_n(\alpha)}{V_n(\alpha)}$$

where $U_n(\alpha)$ and $V_n(\alpha)$ are polynomials in α with integral coefficients. By hypothesis

$$\frac{U_n(\alpha)}{V_n(\alpha)} = \frac{P^*(n,\alpha)}{Q^*(n,\alpha)} ,$$

and

$$U_n(\alpha)Q^*(n,\alpha) = V_n(\alpha)P^*(n,\alpha)$$

$$= P^*(n,\alpha) \prod_{k=1}^{n} (n+k+\alpha).$$

Since α is transcendental, the only way this can hold is if it holds for all coefficients of the various powers of α independently. That is, we must have

$$U_n(x)Q^*(n,x) = P^*(n,x) \prod_{k=1}^{n} (n+k+x)$$

identically in x. But since

$$U_n(x) = \sum_{k=1}^{n} \prod_{j \neq k} (n+j+x)$$

then (n+k+x) does not divide $U_n(x)$ for any k, $1 \leq k \leq n$. Hence, we must have

$$(n+k+x) \mid Q^*(n,x) \quad \text{for all } k, \quad 1 \leq k \leq n$$

and consequently

$$\prod_{k=1}^{n} (n+k+x) \mid Q^*(n,x).$$

This is impossible, however, since $Q^*(n,x)$ has some maximum degree in x which will be exceeded by the degree of x occurring on the left-hand side by taking n sufficiently large. This completes the proof of the proposition.

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Att.
References

REFERENCES

1. Hardy, G. H. and Wright, E. M., An Introduction to the Theory of Numbers, Oxford (1960), p. 350.
2. Johnson, S. C., Several Interesting Problems, Memorandum for File, July 1, 1970.