COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE— Sequential Generation by Transpositions of all the Arrangements of n Symbols

MM 64 – 1271 – 5
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CASE CHARGED— 39199, 20878-4

FILING CASES— 39199-11
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DATE— June 9, 1964

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FILING SUBJECTS— Permutations
Group Theory
Computer Algorithms

ABSTRACT

It is shown that the set of n! arrangements of n distinct symbols can be generated by successive transpositions of adjacent symbols. The sequence of transpositions can be specified by a sequence of integers $A_n(k)$, $k = 1, \ldots, n!$, where the k^{th} transposition interchanges the symbols in positions $A_n(k)$ and $A_n(k) + 1$. A simple recursive algorithm for computing this sequence is given.

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SUBJECT: Sequential Generation by Transpositions of all the Arrangements of n Symbols - Cases 39199-11 and 20878

DATE: June 9, 1964

FROM: A. J. Goldstein
R. L. Graham
MM-64-1271-5
MM-64-1213-12

MEMORANDUM FOR FILE

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct symbols and let C_n be the set of all n! arrangements of the x_i. The elements of C_n will be written in the form $x_{i_1} x_{i_2} \ldots x_{i_n}$ where the i_k are just the integers 1,2,...,n in some order.

If n = 3, then the sequence $B_3 = (B_3(k): \ 0 \leq k < 6)$ defined by:

$$B_3 = (x_1x_2x_3, \ x_1x_3x_2, \ x_3x_1x_2, \ x_3x_2x_1, \ x_2x_3x_1, \ x_2x_1x_3)$$

has the following properties:

(1) $B_3(k+1)$ can be formed from $B_3(k)$ by the single transposition of two adjacent x_j (where $B_3(k+3!)$ is defined to be $B_3(k)$).

(2) All elements of C_3 occur in B_3 exactly once.

It is the purpose of this note to present a simple algorithm for arranging the elements of C_n into a sequence B_n so that (1) and (2) are satisfied (with 3 replaced by n). In particular the algorithm recursively computes the sequence of integers $A_n = (A_n(k): \ 0 \leq k < n!)$ where $B_n(k+1)$ is obtained from $B_n(k)$ by interchanging the symbols in positions $A_n(k)$ and $A_n(k) + 1$, and $B_n(j+n!) = B_n(j)$.

If the sequence A_n is stored in the computer, then one has a very rapid method of generating permutations.*

It is no restriction to suppose that X_n is the set of integers from 1 to n.

The basic idea we shall use is the following: Suppose B_n has been defined so that (1) and (2) are satisfied. We then construct B_{n+1} by combining slightly altered copies of B_n. Specifically, suppose for each k, $1 \leq k \leq n$, r_k is chosen so that

$$A_n(r_k) < n-1$$

and

$$B_n(r_k) = i_1 \ldots i_{n-1} k$$

and therefore

$$B_n(r_k+1) = j_1 \ldots j_{n-1} k.$$

Certainly such a choice is always possible. Now, for each k and r_k form a sequence $\hat{B}^k_{n+1}$ of n! arrangements of the integers from 1 to n+1 as follows: $\hat{B}^{(k)}_{n+1}$ (0) is obtained from $B_n(r_k)$ by inserting n+1 between its last two elements

$$\hat{B}^{(k)}_{n+1}(0) = i_1 \ldots i_{n-1}(n+1)k.$$

---

*See A. J. Goldstein, "A Computer Oriented Algorithm for Generating Permutations" MM-64-1271-3.

Then $\hat{B}_{n+1}^{(k)}(j)$ for $j = 1,\ldots,n!-1$ is obtained by starting with $\hat{B}_{n+1}^{(k)}(0)$ and "applying the A_n sequence backwards", starting with $A_n(r_k)$. Specifically, $\hat{B}_{n+1}^{(k)}(j+1)$ is obtained from $\hat{B}_{n+1}^{(k)}(j)$ by interchanging the elements in positions $A_n(r_k-j)$ and $A_n(r_k-j)+1$, where $A(i+n!) = A(i)$.

The last copy of B_n, namely $\hat{B}_{n+1}^{(n+1)}$, is obtained from B_n by adjoining $n+1$ to the right hand end of each element of B_n.

Now construct B_{n+1} by "inserting" $\hat{B}_{n+1}^{(k)}$ for $1 \le k \le n$ into $\hat{B}_{n+1}^{(n+1)}$ as follows:

$$
\begin{array}{rcll}
 & \vdots & & \\
 & & & \}\, A_n(r_k-1) \\
\hat{B}_{n+1}^{(n+1)}(r_k) & = & i_1\ldots i_{n-1}k(n+1) & \\
 & & & \}\, n \\
\hat{B}_{n+1}^{(k)}(0) & = & i_1\ldots i_{n-1}(n+1)k & \\
 & & & \}\, A_n(r_k) \\
 & \vdots & & \quad\vdots \\
 & & & \}\, A_n(r_k-n!+1) \\
\hat{B}_{n+1}^{(k)}(n!-1) & = & j_1\ldots j_{n-1}(n+1)k & \\
 & & & \}\, n \\
\hat{B}_{n+1}^{(n+1)}(r_k+1) & = & j_1\ldots j_{n-1}k(n+1) & \\
 & \vdots & & \quad\vdots
\end{array}
$$

The righthand column is the A_{n+1} sequence. Since by hypothesis $j_1\ldots j_{n-1}k$ can be obtained from $i_1\ldots i_{n-1}k$ by a single transposition of a pair of adjacent j's then each term of B_{n+1} can be obtained from the preceding term of B_{n+1} in the same way.

The obvious problem which arises in the previous construction of A_{n+1} from A_n is the determination of the r_k. (Recall that r_k was defined to be an integer such that $B_n(r_k)$ and $B_n(r_k+1)$ both have k as their final term.) We now show that if A_n is defined suitably, then, to obtain A_{n+1} from A_n, the r_k which we use may be chosen to be the integers $\{n + j\cdot(n-1)!: \ 0 \leq j < n\}$. We define A_3 to be (2,1,2,1,2,1) and A_4 to be (2,3,2,3,2,1,2,1,2,3,2,3,2,1,2,1,2,3,2,3,2,1,2,1). (The reader may check that the corresponding B_3 and B_4 satisfy (1) and (2).) Now assume that A_n, $n \geq 4$, has been defined so that it has the structure:

$$A_n = (Y_1, n-1, X_{i_1}, n-1, Y_2, n-1, X_{i_2}, n-1, \ldots$$

$$\ldots, n-1, X_{i_{n-1}}, n-1, Y_n).$$

$\left\{\begin{matrix} X_{i_k} \\ Y_j \end{matrix}\right.$ denotes a <u>block</u> of length $\left\{\begin{matrix} (n-1)! - 1 \\ (n-2)! - 1 \end{matrix}\right.$ for $1 \leq k < n$ and $1 < j < n$. The lengths of Y_1 and Y_n sum to $(n-2)! - 1$. No term of X_{i_k} or Y_j is equal to n-1.

Let $A_n(s)$ denote the first term of X_{i_1} and consider the terms $a_j = A(s + j(n-1)!)$ of A_n for $0 \leq j < n$. Claim: Each X_{i_k} contains exactly one a_j, a_j is never n-1, and exactly one a_j lies in a Y_i. Since each X_{i_k} contains $(n-1)!-1$ terms, no X_{i_k} can contain more than on a_j or less than one a_j, i.e., each X_{i_k} must contain exactly one a_j. If $n \geq 4$, then

$$(n-2)! \equiv 0 \pmod 2.$$

By definition

$$A_n(s-1) = n-1.$$

Thus if

$$A_n(j) = n-1$$

then it follows from consideration of the lengths of X_{i_k} and Y_i that

$$j \equiv s-1 \pmod 2$$

and hence

$$a_j \neq n-1.$$

There are n-1 of the X_{i_k} each of which contains exactly one a_j and no a_j is equal to n-1. Hence exactly one of the a_j falls into a Y_j for some j.

By considering the way in which B_n is related to A_n we see that if $F(B_n(j))$ denotes the final term of $B_n(j)$ (i.e., $F(i_1 \ldots i_n) = i_n$) and $A_n(j) \in X_{i_r}$, $A_n(k) \in X_{i_s}$, then

$$n \neq F(B_n(j)) = F(B_n(j+1)) \neq F(B_n(k)) = F(B_n(k+1)) \neq n.$$

Similarly if

$$A_n(m) \in Y_t,$$

then

$$F(B_n(m)) = F(B_n(m+1)) = n.$$

Thus we can use the a_j, $0 \leq j < n$ for an allowable set of r_k with which to form B_{n+1} from B_n and hence A_{n+1} from A_n. It is not difficult to see that by defining A_{n+1} in this regular manner, A_{n+1} will have a structure similar to that of A_n (with n replaced by n+1) and therefore suitable for defining A_{n+2}, etc. A small amount of computation shows that the following recursive definition for A_n describes the preceding algorithm:†

$$A_3(k) = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 2 & \text{otherwise} \end{cases}$$

$$A_{n+1}(k) = \begin{cases} n & \text{if} \quad k^* = 0 \text{ or } n! \\ A_n\left((n-1)!\left[\frac{k-n}{n^*}\right] + n - k^*\right) & \text{if} \quad k^* < n! \\ A_n\left((n-1)!\left[\frac{k-n}{n^*}\right] + n + k^*\right) & \text{if} \quad k^* > n! \end{cases}$$

for $n \geq 3$ where $A_r(x) = A_r(x+r!)$, $n^* = n! + (n-1)!$, $k^* \equiv k - n \pmod{n^*}$ such that $0 \leq k^* < n^*$ and $[x]$ denotes the greatest integer not exceeding x.

Example: A_5.

If A_5 is generated according to the text we find that

---

†For n = 3 this definition chooses the first term of X_{i_2} instead of X_{i_1} for $A_n(s)$.

$$A_5 = (2,3,2,3,4,3,2,3,2,1,2,1,2,3,2,3,2,1,2,1 \\ 2,3,2,3,2,1,2,1,4,1,2,1,2,3,4,3,2,1,2,1 \\ 2,3,2,3,2,1,2,1,2,3,2,3,2,1,2,1,2,3,4,3 \\ 2,1,2,1,4,1,2,1,2,3,2,3,2,1,2,1,2,3,2,3 \\ 2,1,2,1,2,3,2,3,4,3,2,3,2,1,4,1,2,3,2,3 \\ 2,1,2,1,2,3,2,3,2,1,2,1,2,3,2,3,2,1,4,1)$$

Suppose we wish to calculate $A_5(97)$. Here we have:

$$n = 4$$

$$n^* = 4! + 3! = 30$$

$$k = 97$$

$$k^* \equiv 97 - 4 \equiv 93 \equiv 3 \pmod{30}$$

$$\therefore\ k^* = 3 < 24 = 4!$$

$$\left[\frac{k-n}{n^*}\right] = 3$$

$$\therefore\ A_5(97) = A_4(6 \cdot 3+4-3) = A_4(19).$$

To get $A_4(19)$ we have:

$$n = 3$$

$$n^* = 8$$

$$k = 19$$

$$k^* = 19 - 3 = 16 \equiv 0 \pmod 8$$

$$\therefore\ k^* = 0$$

$$\therefore A_4(19) = n = 3$$

$$\therefore A_5(97) = 3$$

which may be verified directly by examination of the table.

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