Sets of reals with few m-tuple sums

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Abstract

In this note we investigate the following problem.[1] Let k, m and n be positive integers. We determine the largest number $N = N(n, m, k)$ of m-tuples that any n-set X of real numbers can have so that the component sums of these m-tuples take on at most k different values. In particular, we show that the only sets X which achieve this bound are arithmetic progressions.

1 Introduction

A finite ranked partially ordered set (= poset) P is said to be *Sperner* if no antichain (i.e., set of mutually incomparable elements) has larger cardinality than the largest rank, and is *strongly Sperner* if no union of k antichains is larger than the union of the k largest ranks (e.g., see [1]). Further, P is said to be a *Peck* poset if it is strongly Sperner, rank symmetric and rank unimodal. Answering a question of Peck (see [2]), it was shown by Proctor, Saks and Sturtevant (see [3]) that the Cartesian product of two Peck posets (with the usual induced lexicographic partial order) is also a Peck poset.

For positive integers m and n, define $[n] = \{0, 1, \ldots, n-1\}$ and let $([n]^m, \prec)$ denote the poset of m-tuples of $[n]$ partially ordered lexicographically by $\prec$.

[1]suggested by a related problem in a forthcoming preprint of Diaconis, Shao and Soundararajan.

In other words, for $\overline{r} = (r_1, r_2, \ldots, r_m)$ and $\overline{s} = (s_1, s_2, \ldots, s_m)$ in $[n]^m$, we have $\overline{r} \prec \overline{s}$ if and only if $r_i \leq s_i$ for $1 \leq i \leq m$, and for at least one index i, we have $r_i < s_i$.
For a given n-set $X = \{x_1 < x_2 < \ldots x_n\}$ of real numbers and for $(x_{r_1}, x_{r_2}, \ldots, x_{r_m}) \in X^m$, we use the abbreviation $x_{\overline{r}} = (x_{r_1}, x_{r_2}, \ldots, x_{r_m})$ where $\overline{r} = (r_1, r_2, \ldots, r_m) \in [n]^m$. Let $\prec$ denote the partial order on X^m induced by the partial order on the poset $([n]^m, \prec)$. Thus, $(X^m, \prec)$ and $([n]^m, \prec)$ are isomorphic as ranked posets. If neither $x_{\overline{r}} \prec x_{\overline{s}}$ nor $x_{\overline{s}} \prec x_{\overline{r}}$ (and $\overline{r} \neq \overline{s}$) then we say that $x_{\overline{r}}$ and $x_{\overline{s}}$ are *incomparable* in $(X^m, \prec)$. Define $\sigma(x_{\overline{r}}) = \sum_{i=1}^{m} x_{r_i}$.

Fact. If $\sigma(x_{\overline{r}}) = \sigma(x_{\overline{s}})$ and $x_{\overline{r}} \neq x_{\overline{s}}$ then $x_{\overline{r}}$ and $x_{\overline{r}}$ are incomparable.

Proof. Follows from the definitions.

Suppose $R \subseteq [n]^m$ has the property that for all $\overline{r} \in R$, all the $\sigma(x_{\overline{r}})$ are equal. Then by the Fact, this set of $x_{\overline{r}}$ must form an antichain in $(X^m, \prec)$. Consequently, R must form an antichain in $([n]^m, \prec)$. The same argument applies to the situation in which we assume that the sums $\sigma(x_{\overline{r}}), \overline{r} \in R$, assume at most k different values. In this case, since the linearly ordered set $([n], <)$ is clearly a Peck poset, then by results of [3], the product poset $([n]^m, \prec)$ is also a Peck poset, and consequently, the size of R is equal to the sum of the largest k ranks of $([n]^m, \prec)$.

Since the rank of the element $(r_1, r_2, \ldots, r_m) \in ([n]^m, \prec)$ is just $r_1 + r_2 + \ldots + r_m$ then the sizes of the ranks in $([n]^m, \prec)$ are given by the coefficients of the polynomial $(\frac{1-x^{n+1}}{1-x})^m$. Thus, the sum of the k largest ranks of $([n]^m, \prec)$ is just the sum of the k middle coefficients of $(\frac{1-x^{n+1}}{1-x})^m$. In order for a subset $X_R = \{x_{\overline{r}} : \overline{r} = (r_1, r_2, \ldots, r_m) \in R\} \subseteq X^m$ to have this size, then it must be the case that for each of the k largest rank values, say t, *all* of the sums $\sigma(x_{\overline{r}})$ with $r_1 + r_2 + \ldots r_m = t$ must be equal. From this, an easy argument shows that all the differences $x_{i+1} - x_i$ must be equal. Thus, the only sets $X = \{x_1 < x_2 < \ldots < x_n\}$ for which X^m can have this maximum number of elements having only k different component sums are arithmetic progressions.

We summarize the preceding remarks as:

Theorem. Let m, n and k be positive integers. For any n-set $X = \{x_1 < x_2 < \ldots < x_n\}$, the size $N(n, m, k)$ of the largest subset of X^m having only k

different component-wise sums is just the sum of the k middle coefficients of $(\frac{1-x^{n+1}}{1-x})^m$. The only sets X which achieve this bound are arithmetic progressions, in which case the elements of X^m which have these component-wise sums are $(x_{r_1}, x_{r_2}, \ldots, x_{r_m})$ where $r_1 + r_2 + \ldots + r_m = t$ and t is one of the k middle coefficients of $(\frac{1-x^{n+1}}{1-x})^m$.

Remark The above result is similar in spirit to a classic theorem of Stanley which asserts the following.

Theorem [4] Let A be a set of n distinct real numbers, and let $B_1, B_2, \ldots, B_r$ be subsets of A whose element sums take on at most k distinct values. Let $\nu = \lfloor \frac{n-1}{2} \rfloor$ and $\pi = \lfloor \frac{n}{2} \rfloor$. Then r does not exceed the sum of the k middle coefficients of the polynomial

$$2(1+q)(1+q^2)\cdots(1+q^\nu)\cdot(1+q)(1+q^2)\cdots(1+q^\pi).$$

Moreover, this value of r is achieved by choosing $A = \{-\nu, -\nu+1, \ldots, \pi\}$.

References

[1] J. Griggs, On chains and Sperner k-families in ranked posets, *J. Combin. Th (A)*, **23**, (1980), 156–168.

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[4] R. P. Stanley, Weyl groups, the hard Lefschetz theorem and the Sperner property, *SIAM J. Alg. Disc. Methods*, **1**, (1980), 168–184.