# 12 EUCLIDEAN RAMSEY THEORY

R.L. Graham

## INTRODUCTION

Ramsey theory typically deals with problems of the following type. We are given a set $S$, a family $\mathcal{F}$ of subsets of $S$, and a positive integer $r$. We would like to decide whether or not for every partition of $S = C_1 \cup \cdots \cup C_r$ into $r$ subsets, it is always true that some $C_i$ contains some $F \in \mathcal{F}$. If so, we abbreviate this by writing $S \xrightarrow{r} \mathcal{F}$ (and we say $S$ is $r$-Ramsey). If not, we write $S \overset{r}{\not\longrightarrow} \mathcal{F}$. (For a comprehensive treatment of Ramsey theory, see [GRS90].)

In Euclidean Ramsey theory, $S$ is usually taken to be the set of points in some Euclidean space $\mathbb{E}^N$, and the sets in $\mathcal{F}$ are determined by various geometric considerations. The case most studied is the one in which $\mathcal{F} = \text{Cong}(X)$ consists of all *congruent* copies of a fixed finite configuration $X \subset S = \mathbb{E}^N$. In other words, $\text{Cong}(X) = \{gX \mid g \in SO(N)\}$, where $SO(N)$ denotes the special orthogonal group acting on $\mathbb{E}^N$.

Further, we say that $X$ is *Ramsey* if, for all $r$, $\mathbb{E}^N \xrightarrow{r} \text{Cong}(X)$ holds provided $N$ is sufficiently large (depending on $X$ and $r$). This we indicate by writing $\mathbb{E}^N \longrightarrow X$.

Another important case we will discuss (in Section 12.4) is that in which $\mathcal{F} = \text{Hom}(X)$ consists of all *homothetic* copies $aX + \bar{t}$ of $X$, where $a$ is a positive real and $\bar{t} \in \mathbb{E}^N$. Thus, in this case $\mathcal{F}$ is just the set of all images of $X$ under the group of positive homotheties acting on $\mathbb{E}^N$.

It is easy to see that any Ramsey (or $r$-Ramsey) set must be finite. A standard compactness argument shows that if $\mathbb{E}^N \xrightarrow{r} X$ then there is always a *finite* set $Y \subseteq \mathbb{E}^N$ such that $Y \xrightarrow{r} X$. Also, if $X$ is Ramsey (or $r$-Ramsey) then so is any homothetic copy $aX + \bar{t}$ of $X$.

## GLOSSARY

$\mathbb{E}^N \xrightarrow{r}$ **Cong** $(X)$: For any partition $\mathbb{E}^N = C_1 \cup \cdots \cup C_r$, some $C_i$ contains a set congruent to $X$. We say that $X$ is ***r-Ramsey***. When $\text{Cong}(X)$ is understood we will usually write $\mathbb{E}^N \xrightarrow{r} X$.

$\mathbb{E}^N \longrightarrow X$: For every $r$, $\mathbb{E}^N \xrightarrow{r} \text{Cong}(X)$ holds, provided $N$ is sufficiently large. We say in this case that $X$ is ***Ramsey***.

## 12.1 $r$-RAMSEY SETS

In this section we focus on low-dimensional $r$-Ramsey results. We begin by stating three conjectures.

### CONJECTURE 12.1.1

*For any nonequilateral triangle $T$ (i.e., the set of 3 vertices of $T$),*

$$\mathbb{E}^2 \xrightarrow{2} T.$$

### CONJECTURE 12.1.2 (stronger)

*For any partition $\mathbb{E}^2 = C_1 \cup C_2$, every triangle occurs (up to congruence) in $C_1$, or else the same holds for $C_2$, with the possible exception of a single equilateral triangle.*

The partition $\mathbb{E}^2 = C_1 \cup C_2$ with

$$\begin{aligned} C_1 &= \{(x, y) \mid -\infty < x < \infty, 2m \leq y < 2m+1, m = 0, \pm1, \pm2, \ldots\} \\ C_2 &= \mathbb{E}^2 \setminus C_1 \end{aligned}$$

into alternating half-open strips of width 1 prevents the equilateral triangle of side $\sqrt{3}$ from occurring in a single $C_i$. In fact, there are other ways of 2-coloring the plane so as to avoid a monochromatic unit equilateral triangle, such as the so-called "zebra-like" colorings as described in [JKS+09]. It is also shown in [JKS+09] that if the plane is decomposed into the union of an open set and a closed set, then every equilateral triangle occurs at least one of these sets.

### CONJECTURE 12.1.3

*For any triangle $T$,*

$$\mathbb{E}^2 \overset{3}{\not\longrightarrow} T.$$

In the positive direction, we have [EGM+75b]:

### THEOREM 12.1.4

(a) $\mathbb{E}^2 \xrightarrow{2} T$ *if $T$ is a triangle satisfying:*

- (i) *$T$ has a ratio between two sides equal to $2\sin\theta/2$ with $\theta = 30°$, $72°$, $90°$, or $120°$*
- (ii) *$T$ has a $30°$, $90°$, or $150°$ angle* [Sha76]
- (iii) *$T$ has angles $(\alpha, 2\alpha, 180° - 3\alpha)$ with $0 < \alpha < 60°$*
- (iv) *$T$ has angles $(180° - \alpha, 180° - 2\alpha, 3\alpha - 180°)$ with $60° < \alpha < 90°$*
- (v) *$T$ is the degenerate triangle $(a, 2a, 3a)$*
- (vi) *$T$ has sides $(a, b, c)$ satisfying*

$$a^6 - 2a^4b^2 + a^2b^4 - 3a^2b^2c^2 + b^2c^2 = 0$$

*or*

$$a^4c^2 + b^4a^2 + c^4b^2 - 5a^2b^2c^2 = 0$$

- (vii) *$T$ has sides $(a, b, c)$ satisfying*

$$c^2 = a^2 + 2b^2 \quad \textit{with} \quad a < 2b \qquad \text{[Sha76]}$$

- (viii) *$T$ has sides $(a, b, c)$ satisfying*

$$a^2 + c^2 = 4b^2 \ \textit{with} \ \ 3b^2 < 2a^2 < 5b^2 \qquad \text{[Sha76]}$$

(ix) *$T$ has sides equal in length to the sides and circumradius of an isosceles triangle;*

(b) $\mathbb{E}^3 \xrightarrow{2} T$ *for any nondegenerate triangle* $T$

(c) $\mathbb{E}^3 \xrightarrow{3} T$ *for any nondegenerate right triangle* $T$ [BT96]

(d) $\mathbb{E}^3 \stackrel{12}{\not\rightarrow} T$, *a triangle with angles* $(30°, 60°, 90°)$ [Bón93]

(e) $\mathbb{E}^2 \stackrel{2}{\not\rightarrow} Q^2$ *(4 points forming a square)*

(f) $\mathbb{E}^4 \stackrel{2}{\not\rightarrow} Q^2$ [Can96a]

(g) $\mathbb{E}^5 \xrightarrow{2} R^2$, *any rectangle* [Tót96]

(h) $\mathbb{E}^n \stackrel{4}{\not\rightarrow}$ *for any $n$ (a degenerate $(1,1,2)$ triangle)*

(i) $\mathbb{E}^n \stackrel{16}{\not\rightarrow}$ *for any $n$ (a degenerate $(a,b,a+b)$ triangle).*

It is not known whether the 4 in (h) or the 16 in (i) can be replaced by smaller values. Other results of this type can be found in [EGM+73], [EGM+75a], [EGM+75b], [Sha76], and [CFG91].

The 2-point set $X_2$ consisting of two points a unit distance apart is the simplest set about which such questions can be asked, and has a particularly interesting history (see [Soi91] for details). It is clear that

$$\mathbb{E}^1 \stackrel{2}{\not\rightarrow} X_2 \quad \text{and} \quad \mathbb{E}^2 \xrightarrow{2} X_2.$$

To see that $\mathbb{E}^2 \xrightarrow{3} X_2$, consider the 7-point Moser graph shown in Figure 12.1.1. All edges have length 1. On the other hand, $\mathbb{E}^2 \stackrel{7}{\not\rightarrow} X_2$, which can be seen by an appropriate periodic 7-coloring (= partition into 7 parts) of a tiling of $\mathbb{E}^2$ by regular hexagons of diameter 0.9 (see Figure 1.3.1).

FIGURE 12.1.1
*The Moser graph.*

**Definition:** The ***chromatic number*** of $\mathbb{E}^n$, denoted by $\chi(\mathbb{E}^n)$, is the least $m$ such that $\mathbb{E}^n \stackrel{m}{\not\rightarrow} X_2$.

By the above remarks,

$$4 \le \chi(\mathbb{E}^2) \le 7.$$

These bounds have remained unchanged for over 50 years.

Some evidence that $\chi(\mathbb{E}^2) \geq 5$ (in the author's opinion) is given by the following result of O'Donnell:

**THEOREM 12.1.5** [O'D00a], [O'D00b]

*For any $g > 0$, there is 4-chromatic unit distance graph in $\mathbb{E}^2$ with girth greater than $g$.*

Note that the Moser graph has girth 3.

**PROBLEM 12.1.6**

*Determine the exact value of $\chi(\mathbb{E}^2)$.*

The best bounds currently known for $\mathbb{E}^n$ are:

$$(1.239 + o(1))^n < \chi(\mathbb{E}^n) < (3 + o(1))^n$$

(see [FW81], [CFG91],[Rai00], [BMP05]).

A "near miss" for showing $\chi(\mathbb{E}^2) < 7$ was found by Soifer [Soi92]. He shows that there exists a partition $\mathbb{E}^2 = C_1 \cup \cdots \cup C_7$ where $C_i$ contains no pair of points at distance 1 for $1 \leq i \leq 6$, while $C_7$ has no pair at distance $1/\sqrt{5}$.

The best bounds known for $\chi(\mathbb{E}^3)$ are:

$$6 \leq \chi(\mathbb{E}^3) \leq 15.$$

The lower bound is due to Nechushtan [Nec00] and the the upper bound is due (independently to Coulson [Col02], and R. Radoicic and G. Tóth [RT02] (improving earlier results of Székely/Wormald [SW89] and Bóna/Tóth [BT96]).

See Section 1.3 for more details.

An interesting phenomenon, first pointed out by Székely [Szé84], suggests that the true value of $\chi(\mathbb{E}^2)$ may depend on which axioms for set theory are being used. In ZFC, the standard Zermelo/Fraenkel axioms together with the Axiom of Choice, non-(Lebesgue)-measurable sets exist and can be used to prevent monochromatic configurations from occurring. Indeed, it was shown by Falconer [Fal81] that if the plane is decomposed into four *Lebesgue measurable* sets, then one of the sets must contain a unit distance. In other words, the "measurable" chromatic number of the plane is at least 5. On the other hand, if the Axiom of Choice is replaced by the axiom LM which asserts that every set of reals is Lebesgue measurable, then such constructions are not possible and the chromatic number of the plane may be 4 in these systems. (It is known by a result of Solovay [Sol70] that ZFC and ZF + LM are equally consistent). Further results of this type are given in the papers of Shelah and Soifer [SS03, SS04, Soi05]. Sets for which the chromatic number depends on whether or not the color classes are required to be measurable, are said to have an "ambiguous" chromatic number. In [Pay09], Payne constructs a number of interesting examples of unit-distance graphs in $\mathbb{R}^n$ which have ambiguous chromatic number. This is further evidence that the chromatic number of various configurations in $\mathbb{R}^n$ may depend on the flavor of set theory you prefer!

## 12.2 RAMSEY SETS

Recall that $X$ is Ramsey (written $\mathbb{E}^N \longrightarrow X$) if, for all $r$, if $\mathbb{E}^N = C_1 \cup \cdots \cup C_r$ then some $C_i$ must contain a congruent copy of $X$, provided only that $N \geq N_0(X, r)$.

## GLOSSARY

***Spherical:*** $X$ is spherical if it lies on the surface of some sphere.

***Rectangular:*** $X$ is rectangular if it is a subset of the vertices of a rectangular parallelepiped.

***Simplex:*** $X$ is a simplex if it spans $\mathbb{E}^{|X|-1}$.

### THEOREM 12.2.1 [EGM+73]

*If $X$ and $Y$ are Ramsey then so is $X \times Y$.*

Thus, since any 2-point set is Ramsey (for any $r$, consider the unit simplex $S_{2r+1}$ in $\mathbb{E}^{2r}$ scaled appropriately), then so is any rectangular parallelepiped. This implies:

### THEOREM 12.2.2

*Any rectangular set is Ramsey.*

Frankl and Rödl strengthen this significantly in the following way.

**Definition:** A set $A \subset \mathbb{E}^n$ is called ***super-Ramsey*** if there exist positive constants $c$ and $\epsilon$ and subsets $X = X(N) \subset \mathbb{E}^N$ for every $N \geq N_0(X)$ such that:

(i) $|X| < c^n$;

(ii) $|Y| < |X|/(1+\epsilon)^n$ holds for all subsets $Y \subset X$ containing no congruent copy of $A$.

### THEOREM 12.2.3 [FR90]

(i) *All two-element sets are super-Ramsey.*

(ii) *If $A$ and $B$ are super-Ramsey then so is $A \times B$.*

### COROLLARY 12.2.4

*If $X$ is rectangular then $X$ is super-Ramsey.*

In the other direction we have

### THEOREM 12.2.5

*Any Ramsey set is spherical.*

The simplest nonspherical set is the degenerate $(1,1,2)$ triangle.

Concerning simplices, we have the result of Frankl and Rödl:

### THEOREM 12.2.6 [FR90]

*Every simplex is Ramsey.*

In fact, they show that for any simplex $X$, there is a constant $c = c(X)$ such that for all $r$,

$$\mathbb{E}^{c \log r} \xrightarrow{r} X,$$

which follows from their result:

### THEOREM 12.2.7

*Every simplex is super-Ramsey.*

It was an open problem for more than 20 years as to whether the set of vertices of a regular pentagon was Ramsey. This was finally settled by Křiž [Kři91] who proved the following two fundamental results:

**THEOREM 12.2.8** [Kři91]

*Suppose $X \subseteq \mathbb{E}^N$ has a transitive solvable group of isometries. Then $X$ is Ramsey.*

**COROLLARY 12.2.9**

*Any set of vertices of a regular polygon is Ramsey.*

**THEOREM 12.2.10** [Kři91]

*Suppose $X \subseteq \mathbb{E}^N$ has a transitive group of isometries that has a solvable subgroup with at most two orbits. Then $X$ is Ramsey.*

**COROLLARY 12.2.11**

*The vertex sets of the Platonic solids are Ramsey.*

**CONJECTURE 12.2.12**

*Any 4-point subset of a circle is Ramsey.*

Křiž [Kři92] has shown this holds if a pair of opposite sides of the 4-point set are parallel (i.e., form a trapezoid).

Certainly, the outstanding open problem in Euclidean Ramsey theory is to determine the Ramsey sets. The author (bravely?) makes the following:

**CONJECTURE 12.2.13** ($1000)

*Any spherical set is Ramsey.*

If true then this would imply that the Ramsey sets are exactly the spherical sets.

Recently, an alternative conjecture has been suggested by Leader, Russell and Walters [LRW12]. Let us call a finite configuration **C** in Euclidean space *transitive* if it has a transitive group of symmetries. Further, let us say that **C** is *subtransitive* if it is a subset of a transitive configuration.

**CONJECTURE 12.2.14** *[LRW12]*

*Any Ramsey set is subtransitive.*

These authors have also shown [LRW11] that almost all 4-points subsets of a unit circle are *not* subtransitive. Thus, the question as to whether 4-point cyclic subsets are Ramsey sharply separates these two conjectures!

We point out that a result of Spencer [Spe79] shows that any finite configuration $C$ in $\mathbb{E}^n$ is arbitrarily close to a Ramsey set. Let us say that $C'$ is $\epsilon$-close to $C$ if $C'$ can be obtained by moving each point of $C$ by a distance of at most $\epsilon$.

**THEOREM 12.2.15** [Spe79]

*For every finite configuration $C \subset \mathbb{E}^n$ and every $\epsilon > 0$, there is any $\epsilon$-close configuration $C'$ which is a Ramsey set.*

## 12.3 SPHERE-RAMSEY SETS

Since spherical sets play a special role in Euclidean Ramsey theory, it is natural that the following concept arises.

### GLOSSARY

***$S^N(\rho)$*:** A sphere in $\mathbb{E}^N$ with radius $\rho$.

***Sphere-Ramsey:*** $X$ is sphere-Ramsey if, for all $r$, there exist $N = N(X, r)$ and $\rho = \rho(X, r)$ such that

$$S^N(\rho) \xrightarrow{r} X.$$

In this case we write $S^N(\rho) \longrightarrow X$.

For a spherical set $X$, let $\rho(X)$ denote its circumradius, i.e., the radius of the smallest sphere containing $X$ as a subset.

**Remark.** If $X$ and $Y$ are sphere-Ramsey then so is $X \times Y$.

### THEOREM 12.3.1 [Gra83]

*If $X$ is rectangular then $X$ is sphere-Ramsey.*

In [Gra83], it was conjectured that in fact if $X$ is rectangular and $\rho(X) = 1$ then $S^N(1+\epsilon) \longrightarrow X$ should hold. This was proved by Frankl and Rödl [FR90] in a much stronger "super-Ramsey" form.

Concerning simplices, Matousĕk and Rödl proved the following spherical analogue of simplices being Ramsey:

### THEOREM 12.3.2 [MR95]

*For any simplex $X$ with $\rho(X) = 1$, any $r$, and any $\epsilon > 0$, there exists $N = N(X, r, \epsilon)$ such that*

$$S^N(1+\epsilon) \xrightarrow{r} X.$$

The proof uses an interesting mix of techniques from combinatorics, linear algebra, and Banach space theory.

The following results show that the "blowup factor" of $1 + \epsilon$ is really needed.

### THEOREM 12.3.3 [Gra83]

*Let $X = \{x_1, \ldots, x_m\} \subset \mathbb{E}^N$ such that:*

(i) *for some nonempty $I \subseteq \{1, 2, \ldots, m\}$, there exist nonzero $a_i$, $i \in I$, with*

$$\sum_{i \in I} a_i x_i = 0 \in \mathbb{E}^N,$$

(ii) *for all nonempty $J \subseteq I$,*

$$\sum_{j \in J} a_j \neq 0.$$

*Then $X$ is not sphere-Ramsey.*

This implies that $X \subset S^N(1)$ is not sphere-Ramsey if the convex hull of $X$ contains the center of $S^N(1)$.

**Definition:** A simplex $X \subset \mathbb{E}^N$ is called ***exceptional*** if there is a subset $A \subseteq X$, $|A| \geq 2$, such that the affine hull of $A$ translated to the origin has a nontrivial intersection with the linear span of the points of $X \setminus A$ regarded as vectors.

**THEOREM 12.3.4** [MR95]

*If $X$ is a simplex with $\rho(X) = 1$ and $S^N(1) \longrightarrow X$ then $X$ must be exceptional.*

It is not known whether it is true for exceptional $X$ that $S^N(1) \longrightarrow X$. The simplest nontrivial case is for the set of three points $\{a, b, c\}$ lying on some great circle of $S^N(1)$ (with center $o$) so that the line joining $a$ and $b$ is parallel to the line joining $o$ and $c$. We close with a fundamental conjecture:

**CONJECTURE 12.3.5**

*If $X$ is Ramsey, then $X$ is sphere-Ramsey.*

## 12.4 EDGE-RAMSEY SETS

In this variant (introduced in [EGM+75b], we color all the line segments $[a, b]$ in $\mathbb{E}^n$ rather than coloring the points. Analogously to our earlier definition, we will say that a configuration E of line segments is ***edge-Ramsey*** if for any $r$, there is an $N = N(r)$ such any $r$-coloring of the line segments in $\mathbb{E}^N$ contains a monochromatic congruent copy of $E$ (up to some Euclidean motion). The main results known for edge-Ramsey configurations are the following:

**THEOREM 12.4.1** [EGM+75b]

*If $E$ is edge-Ramsey then all edges of $E$ must have the same length.*

**THEOREM 12.4.2** [Gra83]

*If $E$ is edge-Ramsey then the endpoints of the edges of $E$ must lie on two spheres.*

**THEOREM 12.4.3** [Gra83]

*If the endpoints of $E$ do not lie on a sphere and the graph formed by $E$ is not bipartite then $E$ is not edge-Ramsey.*

It is clear that the edge set of an $n$-dimensional simplex is edge-Ramsey. Less obvious (but equally true) are the following.

**THEOREM 12.4.4** [Can96b]

*The edge set of an $n$-cube is edge-Ramsey.*

**THEOREM 12.4.5** [Can96b]

*The edge set of an $n$-dimensional cross polytope is edge-Ramsey.*

This set, a generalization of the octahedron, has as its edges all $2n(n-1)$ line segments of the form $[(0, 0, ..., \pm 1, ..., 0), (0, 0, ..., 0, \pm 1, ..., 0)]$ where the two $\pm 1$'s occur in different positions.

**THEOREM 12.4.6** [Can96b]

*The edge set of a regular $n$-gon is not edge-Ramsey if $n = 5$ or $n \geq 7$.*

Since regular $n$-gons are edge-Ramsey for $n = 2$, 3, and 4, the only undecided value is $n = 6$.

**PROBLEM 12.4.7** *Is the edge set of a regular hexagon edge-Ramsey?*

The situation is not as simple as one might hope since as pointed out by Cantwell [Can96b]:

(i) If $AB$ is a line segment with $C$ as its midpoint, then the set $E_1$ consisting of the line segments $AC$ and $CB$ is not edge-Ramsey, even though its graph is bipartite and $A, B, C$ lie on two spheres.

(ii) There exist nonspherical sets that are edge-Ramsey.

**PROBLEM 12.4.8** *Characterize edge-Ramsey configurations.*

It is not clear at this point what a reasonable conjecture might be. For more results on these topics, see [Can96b] or [Gra83].

## 12.5 HOMOTHETIC RAMSEY SETS AND DENSITY THEOREMS

In this section we will survey various results of the type $\mathbb{E}^N \xrightarrow{r} \mathrm{Hom}(X)$, the set of positive homothetic images $aX + \bar{t}$ of a given set $X$. Thus, we are allowed to dilate and translate $X$ but we cannot rotate it. The classic result of this type is van der Waerden's theorem, which asserts the following:

**THEOREM 12.5.1** [Wae27]

*If $X = \{1, 2, \ldots, m\}$ then $\mathbb{E} \xrightarrow{r} \mathrm{Hom}(X)$.*

(Note that $\mathrm{Hom}(X)$ is just the set of $m$-term arithmetic progressions.)

By the compactness theorem mentioned in the Introduction there exists, for each $m$, a minimum value $W(m)$ such that

$$\{1, 2, \ldots, W(m)\} \xrightarrow{2} \mathrm{Hom}(X).$$

The determination or even estimation of $W(m)$ seems to be extremely difficult. The known values are:

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $W(m)$ | 1 | 3 | 9 | 35 | 178 | 1132 |

The best general result from below (due to Berlekamp—see [GRS90]) is

$$W(p+1) \geq p \cdot 2^p, \quad p \text{ prime}.$$

The best upper bound known follows from a spectacular result of Gowers [Gow01]:

$$W(m) < 2^{2^{2^{2^{m+9}}}}$$

This settled a long-standing \$1000 conjecture of the author. This result is a corollary of Gowers's new quantitative form of Szemerédi's theorem mentioned in the next section. It improves on the earlier bound of Shelah: [She88]:

The following conjecture of the author has been open for more than 30 years:

**CONJECTURE 12.5.2** (\$1000)

*For all $m$,*

$$W(m) \leq 2^{m^2}$$

The generalization to $\mathbb{E}^N$ is due independently to Gallai and Witt (see [GRS90]).

**THEOREM 12.5.3**

*For any finite set $X \subset \mathbb{E}^n$,*

$$\mathbb{E}^N \longrightarrow \mathrm{Hom}(X).$$

We remark here that a number of results in (Euclidean) Ramsey theory have stronger so-called *density* versions. As an example, we state the well-known theorem of Szemerédi.

## GLOSSARY

$\mathbb{N}$: The set of natural numbers $\{1, 2, 3, \ldots\}$.

$\overline{\boldsymbol{\delta}}(\boldsymbol{A})$: The ***upper density*** of a set $A \subseteq \mathbb{N}$ is defined by:

$$\overline{\delta}(A) = \limsup_{n \longrightarrow \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} .$$

**THEOREM 12.5.4** (Szemerédi [Sze75])

*If $A \subseteq \mathbb{N}$ has $\overline{\delta}(A) > 0$ then $A$ contains arbitrarily long arithmetic progressions.*

That is, $A \cap \mathrm{Hom}\{1, 2, \ldots, m\} \neq \emptyset$ for all $m$. This clearly implies van der Waerden's theorem since $\mathbb{N} = C_1 \cup \cdots \cup C_r \Rightarrow \max_i \overline{\delta}(C_i) \geq 1/r$.

Furstenberg [Fur77] has given a quite different proof of Szemerédi's theorem, using tools from ergodic theory and topological dynamics. This approach has proved to be very powerful, allowing Furstenberg, Katznelson, and others to prove density versions of the Hales-Jewett theorem (see [FK91]), the Gallai-Witt theorem, and many others. Gowers has proved the following strong quantitative version of Szemerédi's theorem:

**THEOREM 12.5.5** [Gow01]
*For every $k > 0$, any subset of $1, 2, ..., N$ of size at least $N(\log\log N)^{-c(k)}$ contains a $k$-term arithmetic progression, where $c(k) = 2^{-2^{k+9}}$.*

Recently, the Polymath project, initiated by Gowers, has resulted in several new proofs of the density Hales-Jewett theorem (see [Poly12, Tao09, DKT14]).

There are other ways of expressing the fact that $A$ is relatively dense in $\mathbb{N}$ besides the condition that $\overline{\delta}(A) > 0$. One would expect that these could also be used as a basis for a density version of van der Waerden or Gallai-Witt. Very little is currently known in this direction, however. We conclude this section with several conjectures of this type.

**CONJECTURE 12.5.6** (Erdős)
*If $A \subseteq \mathbb{N}$ satisfies $\sum_{a\in A} 1/a = \infty$ then $A$ contains arbitrarily long arithmetic progressions.*

**CONJECTURE 12.5.7** (Graham)
*If $A \subseteq \mathbb{N}\times\mathbb{N}$ with $\sum_{(x,y)\in A} 1/(x^2+y^2) = \infty$ then $A$ contains the 4 vertices of an axis-aligned square.*

More generally, I expect that $A$ will always contain a homothetic image of $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ for all $m$. Of course, if we assume $A$ has positive upper density, then this result follows from the density Hales-Jewett theorem [FK91]. A nice combinatorial proof by Solymosi for the square appears in [Sol04].

Finally, we mention a direction in which the group $SO(n)$ is enlarged to allow dilations as well.

**Definition:** For a set $W \subseteq \mathbb{E}^k$, define the ***upper density*** $\overline{\delta}(W)$ of $W$ by

$$\overline{\delta}(W) := \limsup_{R\longrightarrow\infty} \frac{m(B(o,R)\cap W)}{m(B(o,R))},$$

where $B(o,R)$ denotes the $k$-ball $\left\{(x_1,\ldots,x_k)\in\mathbb{E}^k \;\middle|\; \sum_{i=1}^k x_i^2 \le R^2\right\}$ centered at the origin, and $m$ denotes Lebesgue measure.

**THEOREM 12.5.8** (Bourgain [Bou86])
*Let $X \subseteq \mathbb{E}^k$ be a simplex. If $W \subseteq \mathbb{E}^k$ with $\overline{\delta}(W) > 0$ then there exists $t_0$ such that for all $t > t_0$, $W$ contains a congruent copy of $tX$.*

Some restrictions on $X$ are necessary as the following result shows.

**THEOREM 12.5.9** (Graham [Gra94])
*Let $X \subseteq \mathbb{E}^k$ be nonspherical. Then for any $N$ there exist a set $W \subseteq \mathbb{E}^N$ with $\overline{\delta}(W) > 0$ and a set $T \subseteq \mathbb{R}$ with $\underline{\delta}(T) > 0$ such that $W$ contains no congruent copy of $tX$ for any $t \in T$.*

Here $\underline{\delta}$ denotes ***lower density***, defined similarly to $\overline{\delta}$ but with $\liminf$ replacing lim sup.

It is clear that much remains to be done here.

## 12.6 VARIATIONS

There are quite a few variants of the preceding topics that have received attention in the literature (e.g., see [Sch93]). We mention some of the more interesting ones.

### ASYMMETRIC RAMSEY THEOREMS

Typical results of this type assert that for given sets $X_1$ and $X_2$ (for example), for every partition of $\mathbb{E}^N = C_1 \cup C_2$, either $C_1$ contains a congruent copy of $X_1$, or $C_2$ contains a congruent copy of $X_2$. We can denote this by

$$\mathbb{E}^N \xrightarrow{2} (X_1, X_2).$$

Here is a sampling of results of this type (more of which can be found in [EGM$^+$73], [EGM$^+$75a], [EGM$^+$75b]).

(i) $\mathbb{E}^2 \xrightarrow{2} (T_2, T_3)$ where $T_i$ is any subset of $\mathbb{E}^2$ with $i$ points, $i = 2, 3$.

(ii) $\mathbb{E}^2 \xrightarrow{2} (P_2, P_4)$ where $P_2$ is a set of two points at a distance 1, and $P_4$ is a set of four collinear points with distance 1 between consecutive points.

(iii) $\mathbb{E}^3 \xrightarrow{2} (T, Q^2)$ where $T$ is an isoceles right triangle and $Q^2$ is a square.

(iv) $\mathbb{E}^2 \xrightarrow{2} (P_2, T_4)$ where $P_2$ is as in (ii) and $T_4$ is any set of four points [Juh79].

(v) There is a set $T_8$ of 8 points such that

$$\mathbb{E}^2 \stackrel{2}{\not\longrightarrow} (P_2, T_8) \quad \text{[CT94]}.$$

This strengthens an earlier result of Juhász [Juh79], which proved this for a certain set of 12 points.

### POLYCHROMATIC RAMSEY THEOREMS

Here, instead of asking for a copy of the target set $X$ in a single $C_i$, we require only that it be contained in the union of a small number of $C_i$, say at most $m$ of the $C_i$.

Let us indicate this by writing $\mathbb{E}^N \underset{m}{\rightarrow} X$.

(i) If $\mathbb{E}^N \underset{m}{\rightarrow} X$ then $X$ must be embeddable on the union of $m$ concentric spheres [EGM$^+$73].

(ii) Suppose $X_i$ is finite and $\mathbb{E}^N \underset{m_i}{\longrightarrow} X_i$, $1 \le i \le t$. Then

$$\mathbb{E}^N \xrightarrow[m_1 m_2 \cdots m_t]{} X_1 \times X_2 \times \cdots \times X_t \qquad \text{[ERS83]}.$$

(iii) If $X_6$ is the 6-point set formed by taking the four vertices of a square together with the midpoints of two adjacent sides then $\mathbb{E}^2 \not\rightarrow X_6$ but $\mathbb{E}^2 \underset{2}{\longrightarrow} X_6$.

(iv) If $X$ is the set of vertices of a regular simplex in $\mathbb{E}^N$ together with the trisection points of each of its edges then

$$\mathbb{E}^2 \not\rightarrow X_6 \quad \text{but} \quad \mathbb{E}^2 \underset{3}{\longrightarrow} X_6.$$

It is not known if $\mathbb{E}^2 \underset{2}{\longrightarrow} X_6$. Many other results of this type can be found in [ERS83].

## PARTITIONS OF $\mathbb{E}^n$ WITH ARBITRARILY MANY PARTS

Since $\mathbb{E}^2 \overset{7}{\not\rightarrow} P_2$, where $P_2$ is a set of two points with unit distance, one might ask whether there is any nontrivial result of the type $\mathbb{E}^2 \overset{m}{\longrightarrow} \mathcal{F}$ when $m$ is allowed to go to infinity. Of course, if $\mathcal{F}$ is sufficiently large, then there certainly are. There are some interesting geometric examples for which $\mathcal{F}$ is not too large.

**THEOREM 12.6.1** [Gra80a]

*For any partition of $\mathbb{E}^n$ into finitely many parts, some part contains, for all $\alpha > 0$ and all sets of lines $L_1, \ldots, L_n$ that span $\mathbb{E}^n$, a simplex having volume $\alpha$ and edges through one vertex parallel to the $L_i$.*

Many other theorems of this type are possible (see [Gra80a]).

## PARTITIONS WITH INFINITELY MANY PARTS

Results of this type tend to have a strong set-theoretic flavor. For example: $\mathbb{E}^2 \overset{\aleph_0}{\not\rightarrow} T_3$ where $T_3$ is an equilateral triangle [Ced69]. In other words, $\mathbb{E}^2$ can be partitioned into countably many parts so that no part contains the vertices of an equilateral triangle. In fact, this was recently strengthened by Schmerl [Sch94b] who showed that for all $N$,

$$\mathbb{E}^N \overset{\aleph_0}{\not\rightarrow} T_3.$$

In fact, this result holds for *any* fixed triangle $T$ in place of $T_3$ [Sch94b]. Schmerl also has shown [Sch94a] that there is a partition of $\mathbb{E}^N$ into countably many parts such that no part contains the vertices of *any* isoceles triangle.

Another result of this type is this:

**THEOREM 12.6.2** [Kun]

*Assuming the Continuum Hypothesis, it is possible to partition $\mathbb{E}^2$ into countably many parts, none of which contains the vertices of a triangle with* **rational** *area.*

We also note the interesting result of Erdős and Komjath:

**THEOREM 12.6.3** [EK90]

*The existence of a partition of $\mathbb{E}^2$ into countably many sets, none of which contains the vertices of a* **right** *triangle is equivalent to the Continuum Hypothesis.*

The reader can consult Komjath [Kom97] for more results of this type.

### COMPLEXITY ISSUES

S. Burr [Bur82] has shown that the algorithmic question of deciding if a given set $X \subset \mathbb{N} \times \mathbb{N}$ can be partitioned $X = C_1 \cup C_2 \cup C_3$ so that $x, y \in C_i$ implies distance$(x, y) \geq 6$, for $i = 1, 2, 3$, is NP-complete. (Also, he shows that a certain infinite version of this is undecidable.)

Finally, we make a few remarks about the celebrated problem of Esther Klein (who became Mrs. Szekeres), which, in some sense, initiated this whole area (see [Sze73] for a charming history).

**THEOREM 12.6.4** [ES35]

*There is a minimum function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that any set of $f(n)$ points in $\mathbb{E}^2$ in general position contains the vertices of a convex $n$-gon.*

This result of Erdős and George Szekeres actually spawned an independent genesis of Ramsey theory. The best bounds currently known for $f(n)$ are:

$$2^{n-2} + 1 \leq f(n) \leq 2^{n+4n^{4/5}}.$$

The lower bound appears in [ES35]. The upper bound is a striking new result of Andrew Suk [Suk16]. It applies for $n$ sufficiently large and is the first significant improvement of the original upper bound $\binom{2n-4}{n-2+1}$ of Erdős and Szekeres.

**CONJECTURE 12.6.5**

*Prove (or disprove) that $f(n) = 2^{n-2} + 1$, $n \geq 3$.*

(See Chapter 1 of this Handbook for more details.)

## 12.7 SOURCES AND RELATED MATERIAL

### SURVEYS

The principal surveys for results in Euclidean Ramsey theory are [GRS90], [Gra80b], [Gra85], and [Gra94]. The first of these is a monograph on Ramsey theory in general, with a section devoted to Euclidean Ramsey theory, while the last three are specifically about the topics discussed in the present chapter.

### RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 13: Geometric discrepancy theory and uniform distribution

### REFERENCES

[BMP05] P. Brass, W.O.J. Moser, and J. Pach. *Research Problems in Discrete Geometry.* Springer, New York, 2005

[Bón93] M. Bóna. A Euclidean Ramsey theorem. *Discrete Math.*, 122:349–352, 1993.

[Bou86] J. Bourgain. A Szemerédi type theorem for sets of positive density in $\mathbb{R}^k$. *Israel J. Math.*, 54:307–316, 1986.

[BT96] M. Bóna and G. Tóth. A Ramsey-type problem on right-angled triangles in space. *Discrete Math.*, 150:61–67, 1996

[Bur82] S.A. Burr. An NP-complete problem in Euclidean Ramsey theory. In *Proc. 13th Southeastern Conf. on Combinatorics, Graph Theory and Computing*, vol. 35, pages 131–138, 1982.

[Can96a] K. Cantwell. Finite Euclidean Ramsey theory. *J. Combin. Theory, Ser. A*, 73:273–285, 1996.

[Can96b] K. Cantwell. Edge-Ramsey theory. *Discrete Comput. Geom.*, 15:341-352, 1996.

[Ced69] J. Ceder. Finite subsets and countable decompositions of Euclidean spaces. *Rev. Roumaine Math. Pures Appl.*, 14:1247–1251, 1969.

[CFG91] H.T. Croft, K.J. Falconer, and R.K. Guy. *Unsolved Problems in Geometry.* Springer-Verlag, New York, 1991.

[Col02] D.A. Coulson. A 15-coloring of 3-space omitting distance one. *Discrete Math.*, 256:83–90, 2002.

[CT94] G. Csizmadia and G. Tóth. Note on a Ramsey-type problem in geometry. *J. Combin. Theory, Ser. A*, 65:302–306, 1994.

[DKT14] P. Dodos, V. Kamellopoulos and K. Tyros. A simple proof of the density Hales-Jewett theorem, *Int. Math Res. Notices* (2014),(12): 3340–3352.

[EGM+73] P. Erdős, R.L. Graham, P. Montgomery, B.L. Rothschild, J. Spencer, and E.G. Straus. Euclidean Ramsey theorems. *J. Combin. Theory, Ser. A*, 14:341–63, 1973.

[EGM+75a] P. Erdős, R.L. Graham, P. Montgomery, B.L. Rothschild, J. Spencer, and E.G. Straus. Euclidean Ramsey theorems II. In A. Hajnal, R. Rado, and V. Sós, editors, *Infinite and Finite Sets I*, pages 529–557, North-Holland, Amsterdam, 1975.

[EGM+75b] P. Erdős, R.L. Graham, P. Montgomery, B.L. Rothschild, J. Spencer, and E.G. Straus. Euclidean Ramsey theorems III. In A. Hajnal, R. Rado, and V. Sós, editors, *Infinite and Finite Sets II*, pages 559–583, North-Holland, Amsterdam, 1975.

[EK90] P. Erdős and P. Komjáth. Countable decompositions of $\mathbb{R}^2$ and $\mathbb{R}^3$. *Discrete Comput. Geom.*, 5:325–331, 1990.

[ERS83] P. Erdős, B. Rothschild, and E.G. Straus. Polychromatic Euclidean Ramsey theorems. *J. Geom.*, 20:28–35, 1983.

[ES35] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compos. Math.*, 2:463–470, 1935.

[Fal81] K.J. Falconer. The realization of distances in measurable subsets covering $RR^n$ *J. Combin. Theory, Ser. A*, 31:184–189, 1981.

[FK91] H. Furstenberg and Y. Katznelson. A density version of the Hales-Jewett theorem. *J. Anal. Math.*, 57:64–119, 1991.

[FR90] P. Frankl and V. Rödl. A partition property of simplices in Euclidean space. *J. Amer. Math. Soc.*, 3:1–7, 1990.

[Fur77] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Anal. Math.*, 31:204–256, 1977.

[FW81] P. Frankl and R.M. Wilson. Intersection theorems with geometric consequences. *Combinatorica*, 1:357–368, 1981.

[Gow01] T. Gowers. A new proof of Szemerédi's theorem. *Geom. Funct. Anal.*, 11:465–588, 2001.

[Gra80a] R.L. Graham. On partitions of $\mathbb{E}^n$. *J. Combin. Theory, Ser. A*, 28:89–97, 1980.

[Gra80b] R.L. Graham. Topics in Euclidean Ramsey theory. In J. Nešetřil and V. Rödl, editors, *Mathematics of Ramsey Theory*, Springer-Verlag, Heidelberg, 1980.

[Gra83] R.L. Graham. Euclidean Ramsey theorems on the $n$-sphere. *J. Graph Theory*, 7:105–114, 1983.

[Gra85] R.L. Graham. Old and new Euclidean Ramsey theorems. In J.E. Goodman, E. Lutwak, J. Malkevitch, and R. Pollack, editors, *Discrete Geometry and Convexity*, vol. 440 of Ann. New York Acad. Sci., pages 20–30, 1985.

[Gra94] R.L. Graham. Recent trends in Euclidean Ramsey theory. *Discrete Math.*, 136:119–127, 1994.

[GRS90] R.L. Graham, B.L. Rothschild, and J. Spencer. *Ramsey Theory*, 2nd edition. Wiley, New York, 1990.

[JKS+09] V. Jelínek, K. Kynčl, R. Stolař, and T. Valla. Monochromatic triangles in two-colored plane. *Combinatorica,* 29:699–718, 2009.

[Juh79] R. Juhász. Ramsey type theorems in the plane. *J. Combin. Theory, Ser. A*, 27:152–160, 1979.

[Kom97] P. Komjáth. Set theory: geometric and real. R.L. Graham and J. Nešetřil, editors, *The mathematics of Paul Erdős, II*, vol. 14 of Algorithms and Combinatorics, pages 461–466, Springer, Berlin, 1997.

[Kři91] I. Kříž. Permutation groups in Euclidean Ramsey theory. *Proc. Amer. Math. Soc.*, 112:899–907, 1991.

[Kři92] I. Kříž. All trapezoids are Ramsey. *Discrete Math.*, 108:59–62, 1992.

[Kun] K. Kunen. Personal communication.

[LRW11] I. Leader, P.A. Russell, and M. Walter. Transitive sets and cyclic quadrilaterals. *J. Combinatorics*, 2:457–462, 2011.

[LRW12] I. Leader, P.A. Russell, and M. Walter. Transitive sets in Euclidean Ramsey theory. *J. Combin. Theory, Ser. A* 119:382–396, 2012.

[MR95] J. Matoušek and V. Rödl. On Ramsey sets on spheres. *J. Combin. Theory, Ser. A*, 70:30–44, 1995.

[Nec00] O. Nechushtan. A note on the space chromatic number. *Discrete Math.*, 256:499–507, 2002.

[O'D00a] P. O'Donnell. Arbitrary girth, 4-chromatic unit distance graphs in the plane; Part 1: Graph Description. *Geombinatorics*, 9:145–150, 2000.

[O'D00b] P. O'Donnell. Arbitrary girth, 4-chromatic unit distance graphs in the plane; Part 2: Graph embedding. *Geombinatorics*, 9:180–193, 2000.

[Pay09] M.S. Payne. Unit distance graphs with ambiguous chromatic number. *Electr. J. Combin.*, 16:N31, 2009.

[Poly12] D. H. J. Polymath. A new proof of the density Hales-Jewett theorem. Ann. Math., 175 (2012), 1283–1327.

[Rai00] A.M. Raigorodskii. On the chromatic number of the space. *Usp. Mat. Nauk*, 55:147–148, 2000.

[RT02] R. Radoičić and G. Tóth. Note on the chromatic number of the space. In B. Aronov et al., editors, *Discrete and Computational Geometry*, vol. 25 of *Algorithms and Combinatorics*, pages 695–698, Springer, Berlin, 2003.

[Sch93] P. Schmitt. Problems in discrete and combinatorial geometry. In P.M. Gruber and J.M. Wills, editors, *Handbook of Convex Geometry*, volume A, North-Holland, Amsterdam, 1993.

[Sch94a] J.H. Schmerl. Personal communication, 1994.

[Sch94b] J.H. Schmerl. Triangle-free partitions of Euclidean space. *Bull. London Math. Soc.*, 26:483–486, 1994.

[Sha76] L. Shader. All right triangles are Ramsey in $\mathbb{E}^2$! *J. Combin. Theory, Ser. A*, 20:385–389, 1976.

[She88] S. Shelah. Primitive recursive bounds for van der Waerden numbers. *J. Amer. Math. Soc.*, 1:683–697, 1988.

[Soi91] A. Soifer. Chromatic number of the plane: A historical survey. *Geombinatorics*, 1:13–14, 1991.

[Soi92] A. Soifer. A six-coloring of the plane. *J. Combin. Theory, Ser. A*, 61:292–294, 1992.

[Soi05] A. Soifer. Axiom of choice and chromatic number of $\mathbb{R}^n$. *J. Combin. Theory, Ser. A*, 110:169–173, 2005.

[Sol04] J. Solymosi. A note on a question of Erdős and Graham. *Combin. Probab. Comput.*, 13:263–267, 2004.

[Sol70] R.M. Solovay. A model of set theory in which every set of reals is Lebesgue measurable. *Ann. of Math.*, 92:1–56, 1970.

[Spe79] J.H. Spencer All finite configurations are almost Ramsey. *J. Combin. Theory, Ser. A*, 27:410–403, 1979.

[SS03] S. Shelah and A. Soifer. Axiom of choice and chromatic number of the plane. *J. Combin. Theory, Ser. A*, 103:387–391, 2003.

[SS04] S. Shelah and A. Soifer. Axiom of choice and chromatic number: example on the plane. *J. Combin. Theory, Ser. A*, 105:359–364, 2004.

[Suk16] A. Suk. On the Erdős-Szekeres convex polygon problem. Preprint, `arXiv:1604.08657`, 2016.

[Sze73] G. Szekeres. A combinatorial problem in geometry: Reminiscences. In J. Spencer, editor, *Paul Erdős: The Art of Counting, Selected Writings*, pages xix–xxii, MIT Press, Cambridge, 1973.

[Sze75] E. Szemerédi. On sets of integers containing no $k$ elements in arithmetic progression. *Acta Arith.*, 27:199–245, 1975.

[Szé84] L.A. Székely. Measurable chromatic number of geometric graphs and set without some distances in Euclidean space. *Combinatorica*, 4:213–218, 1984.

[SW89] L.A. Székely and N. Wormald. Bounds on the measurable chromatic number of $\mathbb{R}^n$. *Discrete Math.*, 75:343–372, 1989.

[Tao09] T. Tao. Polymath1 and three new proofs of the density Hales-Jewett theorem, available at http://terrytao.wordpress.com/.

[Tót96] G. Tóth. A Ramsey-type bound for rectangles. *J. Graph Theory*, 23:53–56, 1996.

[Wae27] B.L. van der Waerden. Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wisk.*, 15:212–216, 1927.

# INDEX OF DEFINED TERMS

# INDEX OF CITED AUTHORS

The name of each author cited in a chapter spares only once with a reference to that chapter; either to its first appearance in the chapter's bibliography, or, if not cited there, to its first appearance in the text of the chapter.