On the discrepancy of circular sequences of reals

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Abstract

In this paper we study a refined measure of the discrepancy of sequences of real numbers in $[0,1]$ on a circle C of circumference 1. Specifically, for a sequence $\mathbf{x} = (x_1, x_2, \ldots)$ in $[0,1]$, define the *discrepancy* $D(\mathbf{x})$ of $\mathbf{x}$ by

$$D(\mathbf{x}) = \inf_{n\geq 1} \inf_{m\geq 1} n \, \|x_m - x_{m+n}\|$$

where $\|x_i - x_j\| = \min\{|x_i - x_j|, 1 - |x_i - x_j|\}$ is the distance between x_i and x_j on C. We show that $\sup_{\mathbf{x}} D(\mathbf{x}) \leq \frac{3-\sqrt{5}}{2}$ and that this bound is achieved, strengthening a conjecture of D. J. Newman.

1 Introduction

A basic question in the study of the distribution of real sequences is the quantitative estimation of the extent by which an arbitrary sequence must deviate from some measure of regularity. This topic has an extensive literature, much of which is surveyed in [1], [2], [4], [9], and [10].

As an example, de Bruijn and Erdős in [6] considered the following measure of irregularity for sequences on a circle C of circumference 1 (see also [12], [13], and [14]).

For $x, y \in C$, define the *distance* $d(x,y)$ between x and y to be $||x-y|| = \min\{|x_i - x_j|, 1 - |x_i - x_j|\}$. Thus, $||x-y||$ is just the length of the shorter arc in C joining x and y. For an infinite sequence $\mathbf{x} = (x_1, x_2, \ldots)$ in C, define

$$\omega(\mathbf{x}) = \liminf_{n\to\infty} \inf_{1\leq i<j\leq n} n \, ||x_i - x_j||.$$

Theorem [6]. For any sequence $\mathbf{x}$ in C,

$$\omega(\mathbf{x}) \leq \frac{1}{\ln 4} = 0.72135\ldots. \tag{1}$$

Furthermore, the bound in (1) is best possible, as shown by taking $x_n = \{\log_2(2n-1)\}$ where $\{z\}$ denotes the fractional part of z.

While this sequence $\mathbf{x}$ is optimal with respect to ω, it is certainly not as spread out as one would think that a well-distributed sequence should be. In

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particular, consecutive points x_n and x_{n+1} are very close together. In this note, we consider a related but much more sensitive measure of discrepancy, suggested by the following conjecture of D. J. Newman (see [7]).

For an infinite sequence $\mathbf{x} = (x_1, x_2, \ldots)$ in $[0, 1]$, define

$$\delta(\mathbf{x}) = \inf_{n \geq 1} \liminf_{m \to \infty} n \, |x_m - x_{m+n}|.$$

Conjecture (D. J. Newman).

$$\sup_{\mathbf{x}} \delta(\mathbf{x}) \leq \frac{1}{\sqrt{5}} = 0.44721 \ldots .$$

This conjecture was motivated (in part) by the fact that $\frac{1}{\sqrt{5}}$ occurs frequently in extremal problems involving discrepancy. For example, it follows from standard results in diophantine approximation (e.g., see [11]), that for any real $\theta \geq 0$,

$$\liminf_{n} n\{n\theta\} \leq \frac{1}{\sqrt{5}}. \tag{2}$$

Furthermore, the constant $\frac{1}{\sqrt{5}}$ cannot be replaced by any smaller constant since (2) holds with equality for $\theta = \frac{1+\sqrt{5}}{2}$, for example.

For an infinite sequence $\mathbf{x} = (x_0, x_1, x_2, \ldots)$, we define the *discrepancy* $D(\mathbf{x})$ of $\mathbf{x}$ by

$$D(\mathbf{x}) = \inf_{n} \inf_{m} \; n \, ||x_m - x_{m+n}||.$$

For this measure of discrepancy, any run of n consecutive terms of $\mathbf{x}$ must be just as well dispersed as the first n terms of $\mathbf{x}$. In this sense, it is a more sensitive measure of irregularity of distribution. A natural question concerning this definition is about the choice of *inf* versus *liminf*. As it turns out, the same bounds hold for both versions and therefore we here adapt the stronger notion. More details are discussed in Section 3.

In this paper, we will prove that Newman's conjecture is valid. In particular, we determine the best possible constant which turns out to be somewhat smaller than $\frac{1}{\sqrt{5}}$.

For the discrepancy of a sequence $\mathbf{x}$ of reals on a circle C of circumference 1, we will prove the following.

Theorem 1. $\sup_{\mathbf{x}} D(\mathbf{x}) = \alpha_0 = \frac{3-\sqrt{5}}{2}$.

Note that $\alpha_0 = \frac{3-\sqrt{5}}{2} = 0.381966 \ldots < \frac{1}{\sqrt{5}} = 0.44721 \ldots$.

The analogous problem of bounding the discrepancy of sequences $\mathbf{x}$ which lie in the *interval* $[0, 1]$ (as opposed to the circle C) was considered by the authors in [5]. For this case, we define the discrepancy $D^*(\mathbf{x}) = \inf_{n} \inf_{m} \; n \, |x_m - x_{m+n}|$. As it turns out, the extremal value of $D^*(\mathbf{x})$ is slightly larger than it is for $D(\mathbf{x})$. In particular, the following result [5] was established.

Theorem 2. *For any sequence of reals* $\mathbf{x} = (x_1, x_2, \ldots)$ *with* $x_i \in [0,1]$*, we have*

$$D^*(\mathbf{x}) = \inf_n \inf_m n|x_{m+n} - x_n| \leq \Big(1 + \sum_{k\geq 1} \frac{1}{F_{2k}}\Big)^{-1}$$
$$= \beta = 0.39441967\ldots$$

where F_k *denotes the* k^{th} *Fibonacci number. Furthermore, this upper bound is sharp. For example, it is achieved by the sequence* $\mathbf{x}^*$ *defined by*

$$x_n^* = \beta \sum_{i\geq 1} \frac{\epsilon_i(n)}{F_{2i}} \tag{3}$$

where the $\epsilon_i(n) \in \{0,1,2\}$ *are determined as follows. Every integer* $n \geq 1$ *has a unique expansion as*

$$n = \sum_{i\geq 1} \epsilon_i(n) F_{2i}$$

where $\epsilon_i(n) \in \{0,1,2\}$ *and if* $\epsilon_i(n) = 2 = \epsilon_j(n)$ *for* $i < j$ *then there is an index* k *with* $i < k < j$ *such that* $\epsilon_k(n) = 0$*.*

Before proceeding we first state several facts concerning Fibonacci numbers that will be needed.

The Fibonacci numbers F_n are given by the recurrence $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

(i)

$$F_n = \frac{1}{\sqrt{5}}(\tau^n - \sigma^n)$$

where $\tau = (1+\sqrt{5})/2$ and $\sigma = (1-\sqrt{5})/2$. Thus $\tau + \sigma = 1$ and $\tau\sigma = -1$.

(ii)

$$\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1.$$

(iii) For all $1 \leq s \leq t$,

$$\frac{1}{F_{2s+1}} < \sum_{i=s+1}^{t} \frac{1}{F_{2i}} + \frac{1}{F_{2t}} - \frac{1}{F_{2t+2}} \leq \frac{1}{F_{2s}} - \frac{1}{F_{2s+2}}.$$

In the remainder of the paper, we will use other various well-known properties of the Fibonacci numbers, all of which are standard and can be found in [8].

2 The main results

In the circle C of circumference 1, we will assume the point $0 \in C$ is located at the rightmost point of C and the positive direction on C proceeds in a counterclockwise direction (so that $\frac{1}{4}$ is at the top of C). For an infinite sequence $\mathbf{x} = (x_0, x_1, x_2, \ldots)$, we will assume without of loss of generality that $x_0 = 0$, and $x_1 \leq \frac{1}{2}$.

Given a finite sequence $\mathbf{x}_n = (x_0 = 0, x_1, x_2, \ldots, x_n)$, we will let $\pi(\mathbf{x}_n)$ denote the sequence specifying the *order* in which the x_i appear as we move around C in a counterclockwise direction starting from 0. For example, if

$$\mathbf{x}_7 = \left(\{\tfrac{3}{8}k\} : 0 \leq k \leq 7\right) = (0, \tfrac{3}{8}, \tfrac{6}{8}, \tfrac{1}{8}, \tfrac{4}{8}, \tfrac{7}{8}, \tfrac{2}{8}, \tfrac{5}{8})$$

then $\pi(\mathbf{x}_7) = \langle 0, 3, 6, 1, 4, 7, 2, 5 \rangle$ (see Figure 1) where use the convention that the sequence in the angle brackets denotes the order in which the x_i are encountered as we go around C in the counterclockwise direction. That is, starting at $x_0 = 0$, we see $x_3 = \frac{1}{8}$, then $x_6 = \frac{2}{8}$, then $x_1 = \frac{3}{8}$, etc. As mentioned previously, $\{x\}$ denotes the fractional part of x.

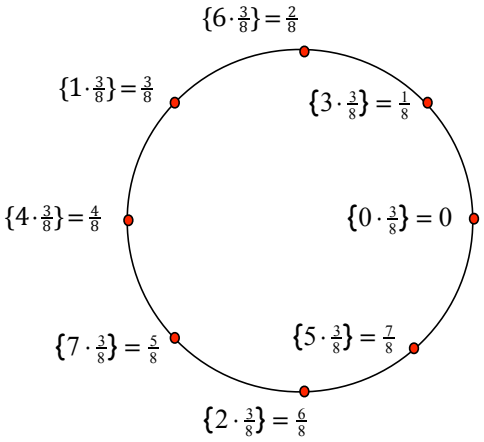

Figure 1: Circle C with the points $\{\frac{3k}{8}\}, 0 \leq k \leq 7$.

Now for a positive integer N, define the restricted discrepancy $D_N(\mathbf{x})$ of $\mathbf{x}$ by

$$D_N(\mathbf{x}) = \inf_{n \leq N} \inf_{m} n\,||x_m - x_{m+n}||.$$

With this restricted discrepancy we will establish the following theorem which will be a step in proving Theorem 1.

Theorem 3. $\sup_{\mathbf{x}} D_N(\mathbf{x}) = \frac{F_{2n-1}}{F_{2n+1}}$ *for* $F_{2n-1} \leq N < F_{2n+1}$.

First, we will show that the bound in Theorem 1 can be achieved.

Define $\bar{\mathbf{y}} = (y_0, y_1, y_2, \ldots)$ with $y_k = \{k\,\alpha_0\}, k \geq 0$ where we recall that $\alpha_0 = \frac{3-\sqrt{5}}{2}$.

Claim. $D(\bar{\mathbf{y}}) = \alpha_0$.

Proof. To begin, observe that we only need to check that $\inf_n n\,||y_n|| \geq \alpha_0$. ($D(\bar{\mathbf{y}})$ is certainly at most α_0 since $||y_1|| = \alpha_0$.) Furthermore, we only have to verify the claim for those y_n which have the property that $||y_n|| < ||y_s||$, for all $s < n$. A quick calculation shows that the order of the first 6 terms of $\bar{y}$ is $\langle 0, 3, 1, 4, 2, 5\rangle$. Thus, extending the order to the next three terms, since y_5 is between y_0 and y_2, then y_6 must be between y_1 and y_3, and y_7 must be between y_2 and y_4. Consequently, the distances $||y_6||$ and $||y_7||$ cannot be smaller than $||y_5||$. However, $||y_8||$ lies between $||y_3||$ and $||y_5||$, and since this interval also contains 0, then $||y_8||$ might be a new champion small distance (in fact, it is). A check shows that it is in the interval between y_0 and y_3. Continuing this argument, we see that y_9 must be between y_1 and y_4, y_{10} must be between y_2 and y_5, y_{11} must be between y_3 and y_6, and y_{12} must be between y_4 and y_7. Thus, none of these 4 points can be closer to 0 than y_8 is. However, y_{13} will lie in the interval defined by y_5 and y_8, and since this interval also contains 0, then $||y_{13}||$ could be a new champion. (Again, it is, and in fact, it lies between y_0 and y_5.) Notice that the indices at which new champions occur are exactly the Fibonacci numbers. This fact is well known and follows from the following calculations.

$$\begin{aligned} F_t\,\alpha_0 - F_{t-2} &= \frac{1}{\sqrt{5}}\left(\frac{1}{\tau^2}(\tau^t - \sigma^t) - (\tau^{t-2} - \sigma^{t-2})\right) \\ &= \frac{1}{\sqrt{5}}(\sigma^{t-2} - \sigma^{t+2}) \\ &= \frac{1}{\sqrt{5}}(1 - \sigma^4)\,\sigma^{t-2} = \sigma^t. \end{aligned}$$

In particular,

$$\begin{aligned} F_{2r}\,\alpha_0 &= F_{2r-2} + \sigma^{2r} > F_{2r-2}, \\ F_{2r+1}\,\alpha_0 &= F_{2r-1} + \sigma^{2r+1} < F_{2r-1}. \end{aligned}$$

This implies that when the time comes for y_{F_t} to be located, it lies between 0 and $y_{F_{t-2}}$ (and not between 0 and $y_{F_{t-1}}$). Therefore, for $t \geq 2$,

$$\begin{aligned} F_t\,||F_t\alpha_0|| &= |F_t\,\sigma^t| \\ &= \frac{1}{\sqrt{5}}(\tau^t\,\sigma^t - \sigma^{2t}) \\ &= \frac{1}{\sqrt{5}}((-1)^t - \sigma^{2t}) \\ &\leq \frac{1}{\sqrt{5}}(1 - \sigma^4) = \alpha_0. \end{aligned}$$

This proves the Claim. □

2.1 Proof of Theorem 1

Suppose $\mathbf{x} = (0, x_1, x_2, \ldots)$ has $D(\mathbf{x}) = \alpha \geq \alpha_0 = 0.381966\ldots$. Thus, we must always have $||x_m - x_{m+n}|| \geq \frac{\alpha}{n}$. First, consider $(0, x_1, x_2)$ (where we have assumed that $x_1 \leq \frac{1}{2}$). If $0 < x_2 < x_1$ then

$$\tfrac{1}{2} \geq ||x_1|| = ||x_2|| + ||x_1 - x_2|| \geq \alpha(\tfrac{1}{2} + 1)$$

so that $\alpha \leq \frac{1}{3}$, which is a contradiction. Hence, $x_2 > x_1$. In other words, the order of these three points is $\langle 0, 1, 2\rangle$. In this case we have

$$1 \geq ||x_1|| + ||x_2|| + ||x_1 - x_2|| \geq ||x_1|| + \alpha(1 + \tfrac{1}{2})$$

so that

$$||x_1|| \leq 1 - \tfrac{3}{2}\alpha \leq 1 - \tfrac{3}{2}\alpha_0 < 0.4271.$$

Of course, the same argument shows that $||x_m - x_{m+1}|| < 0.4271$ for all m.

The next issue is to decide where x_3 can go. If we had the order $\langle 0, 1, 2, 3\rangle$ (i.e., if x_3 were between x_2 and 0), then we would have

$$1 = ||x_1|| + ||x_1 - x_2|| + ||x_2 - x_3|| + ||x_3|| \geq \alpha(1 + 1 + 1 + \tfrac{1}{3}) = \alpha(\tfrac{10}{3})$$

so that

$$\alpha \leq \tfrac{3}{10} < \alpha_0$$

which is a contradiction. Hence, we must have the order $\langle 0, 3, 1, 2\rangle$. In fact, this must be the relative order of any four consecutive terms of $\mathbf{x}$. In particular, since 3 is between 0 and 1, then 4 must be between 1 and 2. Now, where can 5 go? If 5 were between 0 and 3, so that we had the order $\langle 0, 5, 3, 1, 4, 2\rangle$, then

$$1 = ||x_5|| + ||x_5 - x_3|| + ||x_3 - x_4|| + ||x_4 - x_2|| + ||x_2|| \geq \alpha(\tfrac{1}{5} + \tfrac{1}{2} + 1 + \tfrac{1}{2} + \tfrac{1}{2}) = \alpha(\tfrac{27}{10})$$

which implies

$$\alpha \leq \tfrac{10}{27} < 0.3704 < \alpha_0,$$

a contradiction. Hence, we must have the order $\langle 0, 3, 1, 4, 2, 5\rangle$, i.e., 5 is between 2 and 0. Thus, we can conclude that if $D(\mathbf{x}) \geq \alpha_0$ then any six consecutive terms of $\mathbf{x}$ must be in the same order $\langle 0, 3, 1, 4, 2, 5\rangle$ as we go around C in the positive direction.

Here, we will begin to abuse our notation and identify the point x_i with its index i (for typographical convenience). In this case, consider the sequence $\langle 0, 1, 2, 3, 4, 5, 0\rangle$ (which goes around C exactly twice). We then have

$$2 \geq \alpha(1 + 1 + 1 + 1 + 1 + \tfrac{1}{5}) = \tfrac{26}{5}\alpha$$

which implies $\alpha \leq \frac{10}{26} = \frac{5}{13}$.

Let us do two more steps of this process before moving to the general case. Since 5 lies between 2 and 0, then 6 lies between 3 and 1, and 7 lies between 4 and 2. Since none of these intervals contains 0, then no new smaller distance to 0 can occur. However, 8 lies between 5 and 3, and this interval does contain 0. So we need to decide on which side of 0 that 8 could lie. Suppose that 8 were in between 5 and 0. Then consider the sequence $\langle 0, 8, 5, 4, 3, 0\rangle$. Since this goes around C just once, then by our assumption that $D_8(\mathbf{x}) \geq D(\mathbf{x}) = \alpha \geq \alpha_0$ we have,

$$1 \geq \alpha(\tfrac{1}{8} + \tfrac{1}{3} + 1 + 1 + \tfrac{1}{3}) = \tfrac{67}{24}\alpha$$

so that we have $\alpha \leq \frac{24}{67} = 0.3582\ldots < \alpha_0 = 0.3801966\ldots$, a contradiction.

Thus, if $D(\mathbf{x}) \geq \alpha_0$ then 8 must lie between 3 and 0. Further, 9 must lie between 4 and 1, 10 must lie between 5 and 2, 11 must lie between 6 and 3, and 12 must lie between 7 and 4. None of these intervals contain the point 0. However, 13 must lie between 8 and 5, and this interval does contain 0. Again, there are two possibilities for the location of 13. If 13 were between 0 and 8, then consider the sequence $\langle 0, 13, 8, 3, 4, 5, 0\rangle$. This sequence goes around C once, so that we have (since $D_{13}(\mathbf{x}) \geq D(\mathbf{x})$),

$$1 \geq \alpha(\tfrac{1}{13} + \tfrac{1}{5} + \tfrac{1}{5} + 1 + 1 + \tfrac{1}{5}) = \tfrac{174}{65}\alpha.$$

This implies that $\alpha \leq \frac{65}{174} = 0.3736\ldots < \alpha_0$, a contradiction. Hence, 13 must go between 5 and 0 (and the same relative order must hold for any 14 consecutive terms of $\mathbf{x}$).

In general, we must distinguish two cases when a new term is placed into the interval containing the point 0. If follows by induction that this point must be of the form F_t for some t. We have just seen this for the points $F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8$ and $F_7 = 13$. Note that the Fibonacci numbers with *odd* indices occur before 0 and the Fibonacci numbers with *even* indices occur after 0.

The first case is when the new point has the form F_{2n+1}, where we can assume that $n \geq 3$. Then the interval containing 0 is bounded by F_{2n-1} from below and F_{2n} from above. Suppose that F_{2n+1} were between 0 and F_{2n}. Then consider the sequence

$$\langle 0, F_{2n+1}, F_{2n}, F_{2n-2}, F_{2n-2}+1, F_{2n-2}+2, F_{2n-2}+3, \ldots, F_{2n-1}-1, F_{2n-1}, 0\rangle.$$

This path goes around C exactly F_{2n-5} times. Hence, since $D_{F_{2n+1}}(\mathbf{x}) \geq \alpha$, we have

$$\alpha\left(\frac{1}{F_{2n+1}} + \frac{1}{F_{2n-1}} + \frac{1}{F_{2n-1}} + F_{2n-3}\cdot 1 + \frac{1}{F_{2n-1}}\right) \leq F_{2n-5}.$$

This implies that

$$\begin{aligned}\alpha\left(\frac{1}{F_{2n+1}}+\frac{3}{F_{2n-1}}\right) &\leq F_{2n-5}-\alpha F_{2n-3}\\ &\leq F_{2n-5}-\alpha_0 F_{2n-3}\\ &=\frac{1}{\sqrt{5}}\left(\tau^{2n-5}-\sigma^{2n-5}-\frac{1}{\tau^2}(\tau^{2n-3}-\sigma^{2n-3})\right)\\ &=\frac{1}{\sqrt{5}}\left(\sigma^{2n-1}-\sigma^{2n-5}\right)\\ &=\frac{1}{\sqrt{5}}(\sigma^4-1)\sigma_{2n-5}\\ &=-\sigma^{2n-3}.\end{aligned}$$

Thus,

$$\alpha\leq\frac{-\sigma^{2n-3}}{\frac{1}{F_{2n+1}}+\frac{3}{F_{2n-1}}}=\frac{1}{\tau^{2n-3}(\frac{1}{F_{2n+1}}+\frac{3}{F_{2n-1}})}.$$

To get a contradiction, we would like to show

$$\frac{1}{\tau^{2n-3}(\frac{1}{F_{2n+1}}+\frac{3}{F_{2n-1}})}<\alpha_0=\frac{1}{\tau^2},$$

i.e.,

$$\frac{1}{F_{2n+1}}+\frac{3}{F_{2n-1}}>\frac{1}{\tau^{2n-5}}. \tag{4}$$

Now (4) is equivalent to

$$\tau^{2n-5}\left(F_{2n-1}+3F_{2n+1}\right)>F_{2n+1}F_{2n-1},$$

or

$$\begin{aligned}\sqrt{5}\tau^{2n-5}(\tau^{2n-1}-\sigma^{2n-1}+3\tau^{2n+1}-3\sigma^{2n+1}) &\geq (\tau^{2n+1}-\sigma^{2n+1})(\tau^{2n-1}-\sigma^{2n-1})\\ &=\tau^{4n}+\sigma^{4n}-(\tau^2+\sigma^2)\\ &=\tau^{4n}+\sigma^{4n}-3. \qquad (5)\end{aligned}$$

However, (5) holds provided we have

$$\sqrt{5}(3\tau^{4n-4}+\tau^{4n-6}-\sigma^4-3\sigma^6)>\tau^{4n}+\sigma^{4n}-3$$

or

$$(3\sqrt{5}\tau^2+\sqrt{5}-\tau^6)\tau^{4n-6}>\sqrt{5}(\sigma^4+3\sigma^6)+\sigma^{4n}-3. \tag{6}$$

However, this is immediate since

$$(3\sqrt{5}\tau^2 + \sqrt{5} - \tau^6) = \tfrac{3}{2}(\sqrt{5} - 1) = 1.8540\ldots > 0$$

whereas the right-hand side of (6) is less than 0. This shows that F_{2n+1} must go in between F_{2n-1} and 0. In this case, consider the sequence

$$\langle 0, 1, 2, 3, \ldots F_{2n+1} - 1, F_{2n}, 0\rangle.$$

This path goes around C exactly F_{2n-1} times and so we have the inequality

$$\alpha(F_{2n-1} \cdot 1 + \frac{1}{F_{2n+1}}) \leq F_{2n-1}.$$

In this case we deduce that

$$\begin{aligned} D_{F_{2n+1}}(\mathbf{x}) \leq \frac{F_{2n-1}}{F_{2n+1} + \frac{1}{F_{2n+1}}} &= \frac{F_{2n-1}F_{2n+1}}{F_{2n+1}^2 + 1} \\ &= \frac{F_{2n-1}F_{2n+1}}{F_{2n-1}F_{2n+3}} = \frac{F_{2n+1}}{F_{2n+3}}. \end{aligned}$$

The other case is when the new point in question is F_{2n}, a Fibonacci number with an even index. The argument is this case is similar to the preceding case. Here, the interval containing the point 0 is bounded by F_{2n-2} from above and by F_{2n-1} from below. (In other words, as we traverse C in the positive direction, we see the points $F_{2n-1}, 0, F_{2n-2}$ in that order.)

Suppose that F_{2n} is between F_{2n-1} and 0. Then consider the sequence

$$\langle 0, F_{2n}, F_{2n-1}, F_{2n-1} - 1, F_{2n-1} - 2, F_{2n-1} - 3, \ldots, F_{2n-2} + 1, F_{2n-2}, 0\rangle.$$

This covers C exactly F_{2n-5} times. Thus, we have

$$\alpha\left(\frac{1}{F_{2n}} + \frac{2}{F_{2n-2}} + F_{2n-3} \cdot 1\right) \leq F_{2n-5}.$$

Therefore,

$$\begin{aligned} \alpha\left(\frac{1}{F_{2n}} + \frac{2}{F_{2n-2}}\right) &\leq F_{2n-5} - \alpha F_{2n-3} \\ &\leq F_{2n-5} - \alpha_0 F_{2n-3} \\ &= F_{2n-5} - \frac{1}{\tau^2} F_{2n-3} \\ &= \frac{1}{\sqrt{5}}\left(\tau^{2n-5} - \sigma^{2n-5} - \frac{1}{\tau^2}(\tau^{2n-3} - \sigma^{2n-3})\right) \\ &= \frac{1}{\sqrt{5}}\left(\sigma^{2n-1} - \sigma^{2n-5}\right) \\ &= \frac{1}{\sqrt{5}}(\sigma^4 - 1)\sigma^{2n-5} = -\sigma^{2n-3} = \frac{1}{\tau^{2n-3}}. \end{aligned}$$

Thus,

$$\alpha \leq \frac{1}{\tau^{2n-3}\left(\frac{1}{F_{2n}} + \frac{2}{F_{2n-2}}\right)}. \tag{7}$$

So, we will obtain a contradiction to (7) if we can show

$$\frac{1}{\tau^{2n-3}\left(\frac{1}{F_{2n}} + \frac{2}{F_{2n-2}}\right)} < \alpha_0 = \frac{1}{\tau^2},$$

in other words,

$$\frac{1}{F_{2n}} + \frac{2}{F_{2n-2}} > \frac{1}{\tau^{2n-5}}.$$

In fact, it is true that

$$\frac{2}{F_{2n-2}} > \frac{1}{\tau^{2n-5}}.$$

This follows from the fact that

$$\begin{aligned}\frac{F_{2n-2}}{\tau^{2n-5}} &= \frac{1}{\sqrt{5}}\frac{\tau^{2n-2} - \sigma^{2n-2}}{\tau^{2n-5}} \\ &= \frac{1}{\sqrt{5}}(\tau^3 + \sigma^{4n-7}) \\ &< \frac{1}{\sqrt{5}}\tau^3 = 1 + \frac{2}{\sqrt{5}} < 2.\end{aligned}$$

Thus, if $D(\mathbf{x}) \geq \alpha_0$ then F_{2n} must go in between 0 and F_{2n-2}.

Since we have shown that $D(\bar{\mathbf{y}}) = \alpha_0$ for the sequence $\bar{\mathbf{y}} = (y_1, y_2, y_3, \ldots)$ with $y_k = \{k\alpha_0\}$, then the order of the y_k around C must also be in the same as we have deduced here.

Finally, for $N = F_{2n+1}$, we have already noted that

$$D(\mathbf{x}) = \inf_n \inf_m n\,||x_m - x_{m+n}|| \geq \alpha_0$$

implies

$$D_N(\mathbf{x}) = \inf_{n \leq N} \inf_m n\,||x_m - x_{m+n}|| \leq \frac{F_{2n+1}}{F_{2n+3}}.$$

Since $D(\mathbf{x}) \leq D_N(\mathbf{x})$ for all N then

$$\sup_{\mathbf{x}} D(\mathbf{x}) = \lim_{N\to\infty} D_N(\mathbf{x}) = \lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n+3}} = \frac{1}{\tau^2} = \alpha_0.$$

This proves Theorem 1. □

To complete the proof of Theorem 3, it will suffice to exhibit a sequence $\mathbf{x}$ such that

$$D_N(\mathbf{x}) = \frac{F_{2n-1}}{F_{2n+1}} \text{ for } F_{2n-1} \leq N < F_{2n+1}.$$

To do this, we define

$$x_k = \left\{k \cdot \tfrac{F_{2n-1}}{F_{2n+1}}\right\},\ k \geq 0.$$

We need to verify that

$$\inf_{k<F_{2n+1}} \inf_m k\,||x_m - x_{m+k}|| \geq \frac{F_{2n-1}}{F_{2n+1}}. \tag{8}$$

As before, it will suffice to take $m = 0$. Also, it will be enough to show that (8) holds when $k = F_m$ for some m. So we need to prove that

$$\left\|\left\{F_m \cdot \frac{F_{2n-1}}{F_{2n+1}}\right\}\right\| \geq \frac{F_{2n-1}}{F_{2n+1}},\ 1 \leq m < 2n+1. \tag{9}$$

Scaling our problem up by a factor of F_{2n+1}, we have the equivalent condition

$$F_m \cdot (F_m F_{2n-1} (mod^* F_{2n+1})) \geq F_{2n-1},\ 1 \leq m < 2n+1, \tag{10}$$

where $r(mod^* s)$ denotes $min\{r(mod\, s), s - r(mod\, s)\}$. Note that $r(mod^* s) = (s-r)mod^* s$.

For example, for $n = 4, F_7 = 13, F_9 = 34$, we have

m	$F_m \cdot (F_m \cdot 13(mod^* 34))$
1	$1 \cdot 13 = 13 \geq 13$
2	$1 \cdot 13 = 13 \geq 13$
3	$2 \cdot 8 = 16 \geq 13$
4	$3 \cdot 5 = 15 \geq 13$
5	$5 \cdot 3 = 15 \geq 13$
6	$8 \cdot 2 = 16 \geq 13$
7	$13 \cdot 1 = 13 \geq 13$
8	$21 \cdot 1 = 21 \geq 13$

Note that in this example, $F_m \cdot 13(mod^* 34) = F_{9-m}(mod^* 34),\ 1 \leq m \leq 8$. Of course, this is no accident!

We recall a general Fibonacci identity:

$$F_{r+1}F_s - F_r F_{s+1} = (-1)^{s+1} F_{r-s}.$$

Changing variables, this becomes:

$$F_{2n+1}F_{m-1} - F_{2n}F_m = (-1)^m F_{2n+1-m}. \tag{11}$$

However,

$$F_{2n}(mod^* F_{2n+1}) = F_{2n-1}(mod^* F_{2n+1}).$$

Thus,

$$F_m F_{2n} (mod^* F_{2n+1}) = F_m F_{2n-1} (mod^* F_{2n+1}).$$

Therefore, by (11) we have

$$F_m F_{2n} (mod^* F_{2n+1}) = F_{2n+1-m}$$

where the $(-1)^m$ factor has been absorbed into the mod^* function. Thus, (10) becomes

$$F_m F_{2n+1-m} \geq F_{2n-1}, \ 1 \leq m < 2n+1. \tag{12}$$

If fact, we will prove the more general inequality:

$$F_m F_{n-m} \geq F_{n-1}, \ 1 \leq m < n, \ n \geq 2. \tag{13}$$

(13) clearly holds for $n = 2$ and $n = 3$. We prove by induction that it holds for all $n \geq 4$. Since

$$\begin{aligned} F_m F_{n-m} &= F_m (F_{n-1-m} + F_{n-2-m}) \\ &= F_m F_{n-1-m} + F_m F_{n-2-m} \\ &\geq F_{n-2} + F_{n-3} \quad \text{by induction} \\ &= F_{n-1} \end{aligned}$$

then (13) follows, as does (12). This completes the proof of (9) and Theorem 3 is proved. □

3 Concluding remarks.

While we have defined our discrepancy in the form

$$D(\mathbf{x}) = \inf_n \inf_m \, n \, ||x_m - x_{m+n}||,$$

we could have just as well have used the modified version

$$\widehat{D}(\mathbf{x}) = \inf_n \liminf_{m \to \infty} ||x_m - x_{m+n}||.$$

It is not hard to show that the same bound holds for this modified version as well.

The sequence $\mathbf{x}^*$ in (3) which achieves equality in Theorem 2 has curious similarity to the well-known van der Corput sequence $\mathbf{z}$ which arises in the study of the uniform distribution of real sequences (see [1]). Here for each $n \geq 0$, we expand $n = \sum_i \epsilon_i(n) \, 2^i$ into its binary expansion, and then define $z_n = \sum_i \epsilon_i(n) \, 2^{-i}$. That is, we flip the powers of 2 from the numerator to the denominator. This is just what did to form the x_n, namely, we flipped the Fibonacci numbers in the expansion of n from the numerators to the denominators. Is this hinting at some general principal for generating well-distributed sequences?

It would be very interesting to understand the analogous problems for two (or more) dimensions. These questions are well known to be difficult. Even for the case of two-dimensional diophantine approximation (with the sup norm), the value of

$$\gamma = \sup_{\theta,\phi} \liminf_{q} q^{\frac{1}{2}} \max(||q\theta||, ||q\phi||)$$

is not known. It is known that $\gamma \geq \sqrt{\frac{2}{7}}$ (see [3]). This implies that the analogue $D_2(\mathbf{x})$ of $D(\mathbf{x})$ for a sequence $\mathbf{x} \in [0,1] \times [0,1]$ with the sup norm d, namely

$$D_2(\mathbf{x}) = \inf_{n \geq 1} \liminf_{m} n^{\frac{1}{2}}\, d(x_{m+n}, x_m),$$

can remain above $\sqrt{\frac{2}{7}} - \epsilon$ for any $\epsilon > 0$ for a suitable sequence $\mathbf{x}$. This is probably not the best possible value, however. It would be very interesting to know the truth in this case.

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