2.1 The geometry of real valued functions

A function $f$ that takes $n$ inputs and gives $m$ outputs is called vector valued if $m > 1$ and scalar valued if $m = 1$.

We write $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\mathbf{x} \rightarrow f(\mathbf{x})$.

Example: the function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $(x,y,z) \mapsto x^2 + y^2 + z^2$ is scalar valued, whereas

$f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $(x,y,z) \mapsto (x^2 + y^2 + z^2, x + y + z)$

is vector valued.
We can associate to a function $f : \mathbb{C} \rightarrow \mathbb{R}$ a graph.

\[ f : (a, b) \subseteq \mathbb{C} \rightarrow \mathbb{R} \]

Here, the graph is a curve.

\[ f : \mathbb{C} \rightarrow \mathbb{R}^2 \]

Here, the graph is a surface.
2.2 Limits & Continuity

We'll need a couple of definitions before we talk about limits.

**Open set**: Let $U$ be a subset of $\mathbb{R}^n$ (written $U \subset \mathbb{R}^n$). We say that $U$ is an open set if for every $x_0 \in U$ there is some number $r > 0$ such that every point with $\|x - x_0\| < r$ is within $U$.

- $\mathbb{R}$: any interval $(a, b) \subset \mathbb{R}$ is open.

**Intuitively**: $U$ is open when the boundary points of $U$ are not in $U$.

- A point $z \in U$ is on the boundary of $U$ if every neighborhood of $z$ contains at least one point in $U$ and one point not in $U$. 

\[ z \text{ is on the boundary} \]
Limits

Remember that in standard one-dimensional calculus, we used limits to study continuity, define derivatives, improper integrals, ...

We would like to generalize this notion to functions of several variables.

**Definition:** Let $A \subset \mathbb{R}^n$ be an open set and let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$. Let $\bar{z}_0$ be in $A$ or be on the boundary of $A$.

We write $\lim_{x \to \bar{z}_0} f(x) = \bar{b}$ when given any neighborhood $U$ of $\bar{b}$ (i.e., an open set containing $\bar{b}$), $f$ is eventually in $U$ as $\bar{z}$ approaches $\bar{z}_0$.

If $f$ does not approach any vector as $x \to \bar{z}_0$, we say the limit does not exist.
Properties of limits

- If \( \lim_{x \to x_0} f(x) = b_1 \) and \( \lim_{x \to x_0} g(x) = b_2 \) then \( \lim_{x \to x_0} (f(x) + g(x)) = b_1 + b_2 \)

- If \( \lim_{x \to x_0} f(x) = b_1 \) and \( \lim_{x \to x_0} g(x) = b_2 \) then \( \lim_{x \to x_0} c f(x) = c b_1 \)

- When \( m = 1 \) (i.e., \( b_1 \) and \( b_2 \) are scalars) \( \lim_{x \to x_0} f(x) g(x) = b_1 b_2 \)

- When \( f(z) = (f_1(z), f_2(z), \ldots, f_m(z)) \) then \( \lim_{x \to x_0} f(z) = \mathbf{b} = (b_1, b_2, \ldots, b_m) \) if and only if \( \lim_{x \to x_0} f_i(x) = b_i \) for \( i = 1, 2, \ldots, m \).

Examples:
Let \( f : \mathbb{R}^2 \to \mathbb{R} \), \( (x,y) \to x^2 + y^2 \)

Compute \( \lim_{(x,y) \to (0,0)} f(x,y) \)
Solution: \[ \lim_{(x,y) \to (0,0)} f(x,y) = \lim_{(x,y) \to (0,0)} x^2 + y^2 = \lim_{(x,y) \to (0,0)} x^2 + \lim_{(x,y) \to (0,0)} y^2 = 0^2 + 1^2 = 1 \]

Example: Use polar coordinates to find the limit (if it exists)
\[ \lim_{(x,y) \to (0,0)} \frac{7x^2}{x^2 + y^2} \]

Solution: \[ \lim_{(r,\theta) \to (0,0)} \frac{7r^2 \cos^2 \theta}{r^2} = \lim_{r \to 0} 7 \cos^2 \theta = 7 \]

Example: Find the limit or show it doesn't exist
\[ \lim_{(x,y) \to (0,0)} \frac{7x^2}{x^2 + y^2} \]

Solution: To show the limit doesn't exist, we can approach \((0,0)\) from 2 different directions. First, fixing \(y=0\) and letting \(x \to 0\) we get \[ \lim_{(x,y) \to (0,0)} \frac{7x^2}{x^2} = \lim_{x \to 0} 7x^2 = 0 \]
Second, fixing \(x=y\), we get \[ \lim_{(x,y) \to (0,0)} \frac{7x^2}{x^2 + y^2} = \lim_{x \to 0} \frac{7x^2}{2x^2} = \frac{7}{2} \]
Continuous functions:

Definition: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a given function and let $x_0 \in A$. We say that $f$ is continuous at $x_0$ if and only if

$$\lim_{{x \to x_0}} f(x) = f(x_0)$$

If $f$ is continuous at every point in $A$, we say that $f$ is continuous.
Example: \( f(x) = 2x^2 + 3x + 5 \) is continuous
(why?)

Example: \( f(x, y) = xy \) is continuous

because \( \lim_{(x, y) \to (x_0, y_0)} xy = (\lim_{x \to x_0} x)(\lim_{y \to y_0} y) = x_0 y_0 = f(x_0, y_0) \)

for all points \((x_0, y_0)\)

Example: \( f(x, y) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \) is not cont.
(why? Think about \( \lim_{(x, y) \to (0, 0)} f(x, y) \))

Properties: Suppose \( f \) & \( g \) are continuous at \( x_0 \):

\[ f + g \] is also continuous at \( x_0 \)

\[ cf \] is also continuous (\( c \) is a real number)

\[ f + g \] is also continuous at \( x_0 \)

when \( f \) & \( g \) are functions from \( \mathbb{R}^n \) to \( \mathbb{R} \)

\( fg \) is continuous at \( x_0 \)

Let \( f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \), then \( f \) is cont. if and only if the \( f_i \)'s are all cont.
Compositions of continuous functions are continuous:

If \( g \) is continuous at \( x_0 \) and \( f \) is continuous at \( y_0 = g(x_0) \), the \( f \circ g \) is continuous at \( x_0 \).

(Recall that \( (f \circ g)(x) = f(g(x)) \).

2.3 Differentiation:

Partial derivatives

Recall from single variable calculus that we defined for \( f: \mathbb{R} \to \mathbb{R} \), the derivative as

\[
\frac{df}{dx}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{rate of change of } f \text{ as } x \text{ changes}
\]

(when the limit exists)

When we have a function of several variables, i.e.,
\( f: \mathbb{R}^n \to \mathbb{R} \) (for example \( f(x, y) = x^2 + y^2 \)),
we can define the rate of change of \( f \) in each of the \( n \) direction (in our example it would be \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \)).
Let $U \subseteq \mathbb{R}^n$ be open and suppose $f: U \rightarrow \mathbb{R}$

then the partial derivatives of $f$ at the point $(x_1, x_2, \ldots, x_n)$ are defined by

$$\frac{\partial f}{\partial x_1}(x_1, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, x_2 + h, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)}{h}$$

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_j, \ldots, x_n)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}$$

where $e_j = (0, 0, \ldots, 1, 0, \ldots)$

**Example:** Let $f(x, y) = x^2 + xy^3$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy^3) = 2x + y^3$$

(think of $y$ as just a constant)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy^3) = 3xy^2$$

(think of $x$ as just a constant)
Example: \( z = \ln(x^5 + y^4) \)  \[ \frac{\partial z}{\partial x} = \frac{x^5 y^4}{x^5 + y^4} \]  \[ \frac{\partial z}{\partial y} = \frac{y^4 x^5}{x^5 + y^4} \]  (All standard rules of differentiation apply)