Section 3.4

Constrained extrema & Lagrange multipliers

Suppose that you want to minimize the surface area of a can, subject to keeping the volume fixed (for example, to minimize the cost of the materials).

Or, suppose that a particle moves along a curve in a container where the force (or temp., or pressure) is given by some function and we want to find the max/min force that the particle experiences.

In other words, we want to solve

\[ \begin{align*}
\text{maximize} & \quad f(\mathbf{x}) \\
\text{subject to} & \quad g(\mathbf{x}) = c \\
\end{align*} \]

\[ \begin{align*}
\text{e.g.:} & \quad \text{minimize } 2\pi rh + 2\pi r^2 \\
\text{subject to } & \quad \pi r^2 h = 1
\end{align*} \]
Theorem (the method of Lagrange multipliers)

Let \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) be \( C^1 \) functions.

- Let \( x_0 \in U \) and \( g(x_0) = c, \nabla g(x_0) \neq 0 \).
- Let \( S = \text{level set for } g \) with value \( c \):
  \[ S = \{ x \in \mathbb{R}^n \mid g(x) = c \} \]

If \( f \) restricted to \( S \) (denoted \( f|_S \)) has a local max or min, then there is a number \( \lambda \) such that (possibly \( \lambda = 0 \))

\[ \nabla f(x_0) = \lambda \nabla g(x_0) \]

We call \( x_0 \) a critical point of \( f|_S \).
But what does this mean for us?

Suppose that $f$ is continuous and the constraint is closed and bounded.

From the first theorem, we have $
abla f(x_0) = \lambda \nabla g(x_0)$ at local max or min $x_0$. We just need to find the points that satisfy $\nabla f(x) = \lambda \nabla g(x)$ — Lagrange equations (or Lagrange condition).
and check to see if they are minima or maxima or neither

So: Step 1: Write the Lagrange Equations
\[ \nabla f = \lambda \nabla g \]

Step 2: Solve for \( \lambda, x, y, z \)

Step 3: Compute \( f \) at the critical pts. and select the min & max.

Remark: If the constraint is not closed & bounded, the min or max may not exist. E.g. \( f(x, y) = x^2 + y^2 \) s.t. \( x + y = 1 \)

Example: Find the extrema of \( f(x, y) = x^2 + y^2 \) on the circle \( x^2 + y^2 = 1 \)

Solve: \( \nabla f(x, y) = (2x, 2y) \)
\( \nabla g(x, y) = (2x, 2y) \)

Step 2: So we must find \( \lambda \) and \( x \) and \( y \) s.t.
\[ (2x, 2y) = \lambda (2x, 2y) \quad \text{and} \quad x^2 + y^2 = 1 \]
If $x = 0$
\[ x^2 - y^2 = 1 \Rightarrow y = \pm 1 \]
\[ y = -\lambda y \Rightarrow \lambda = -1 \]

If $\lambda = 1$
\[ y = -\lambda y \Rightarrow y = 0 \]
\[ x = \pm 1 \]

So we get the pts: $(0, 1), (0, -1), (1, 0)$
& $(-1, 0)$

**Step 3**
we can now check these to see if they are
maxima or minima

- $\text{Max is } f(1, 0) = f(-1, 0) = 1$
- $\text{Min is } f(0, 1) = f(0, -1) = -1$

**Example**

Find the minimum & maximum of $f(x, y, z) = x + y + z$
subject to $x^2 + y^2 + 3z^2 = 6$

**Solve**
\[ \nabla f(x, y, z) = (1, 1, 1) \]
\[ \nabla g(x, y, z) = (2x, 2y, 6z) \]
Want \( \nabla f = \lambda \nabla g \)

So we want
\[
\begin{align*}
1 &= 2\lambda x \\
1 &= 4\lambda y \\
1 &= 6\lambda z
\end{align*}
\]

4 eqns
4 unknowns

\[ \begin{align*}
x^2 + 2y^2 + 3z^2 &= 6
\end{align*} \]

\( \text{Step 2} \) (Solve for \( x, y, z, \lambda \))

So \( \lambda x = 2\lambda y = 3\lambda z \)

and \( \lambda \neq 0 \) (otherwise \( 1 = 2\lambda x \) is not satisfied)

Thus \( x = 2y = 3z \)

\( \Rightarrow x^2 = 4y^2 = 9z^2 \)

\( \Rightarrow y^2 = \frac{x^2}{4} \) & \( z^2 = \frac{x^2}{9} \)

So \( x^2 + \frac{2x^2}{4} + \frac{3x^2}{9} = 6 \)

\( \Rightarrow \left( \frac{3}{2} + \frac{1}{3} \right)x^2 = 6 \)

\( \Rightarrow \frac{11}{6}x^2 = 6 \Rightarrow x^2 = \frac{36}{11} \)

\( x_1 = \frac{6}{\sqrt{11}} \) \hspace{1cm} x_2 = -\frac{6}{\sqrt{11}}

\( \downarrow \)

By \( \lambda (\times) \)

\( y_1 = \frac{x_1}{2} = \frac{3}{\sqrt{11}} \) \hspace{1cm} y_2 = \frac{x_2}{2} = -\frac{3}{\sqrt{11}}

\( z_1 = \frac{x_1}{3} = \frac{2}{\sqrt{11}} \) \hspace{1cm} z_2 = \frac{x_2}{3} = -\frac{2}{\sqrt{11}}

\( P_1 = \left( \frac{6}{\sqrt{11}}, 3, 2 \right) \) & \( P_2 = \left( -\frac{6}{\sqrt{11}}, -3, -2 \right) \)
Step 3

Finally, checking $P_1$ & $P_2$, we see that $f(P_1) > f(P_2)$. Moreover, our constraint was closed & bounded. So $f(P_1)$ is max and $f(P_2)$ is min.

Chapter 4

Sec 4.1 Acceleration

Recall: Given a path $\mathbf{c}(t) = (x(t), y(t), z(t))$

we can compute $\mathbf{c}'(t) = (x'(t), y'(t), z'(t))$

$V(t)$ Velocity vector

Recall: $\mathbf{c}'(t)$ is the tangent vector to the path at the point $\mathbf{c}(t_0)$.

Recall: $\|\mathbf{c}'(t)\|$ is the speed.
Differentiation rules: $\vec{B}: \mathbb{R} \rightarrow \mathbb{R}^3$, $\vec{C}: \mathbb{R} \rightarrow \mathbb{R}^3$

\[ \frac{d}{dt} (\vec{B}(t) + \vec{C}(t)) = \vec{B}'(t) + \vec{C}'(t) \]

- $\vec{f}(t): \mathbb{R} \rightarrow \mathbb{R}$

\[ (\vec{f}(t) \cdot \vec{C}(t))^\prime = \vec{f}'(t) \cdot \vec{C}(t) + \vec{f}(t) \cdot \vec{C}'(t) \]

\[ \text{e.g.} \quad \vec{f}(t) = t \quad \vec{C}(t) = (t, t^2, t^3) \Rightarrow (\vec{f}(t) \cdot \vec{C}(t))^\prime = (1, 2t, 3t^2) + (t, 2t, 3t^2) = (2t, 3t^2, 4t^3) \]

- $(\vec{B}(t) \times \vec{C}(t))^\prime = \vec{B}'(t) \times \vec{C}(t) + \vec{B}(t) \times \vec{C}'(t)$

- $(\vec{C}(\vec{f}(t)))^\prime = \vec{C}'(\vec{f}(t)) \cdot \vec{f}'(t)$

exercise: prove it
Example: If \( \| \mathbf{c}(t) \| = \text{const} \)

then \( \| \mathbf{\dot{c}}(t) \|^2 = \mathbf{\dot{c}}(t) \cdot \mathbf{\ddot{c}}(t) = \text{const} \).

\[ \Rightarrow (\mathbf{\dot{c}}(t) \cdot \mathbf{\dot{c}}(t))' = 0 \]

\[ \Rightarrow \mathbf{\dot{c}}(t) \cdot \mathbf{\ddot{c}}(t) + \mathbf{\dot{c}}(t) \cdot \mathbf{\ddot{c}}(t) = 0 \]

\[ 2 \mathbf{\dot{c}}(t) \cdot \mathbf{\ddot{c}}(t) = 0 \]

so \( \mathbf{\dot{c}}(t) \) is orthogonal to \( \mathbf{\ddot{c}}(t) \) for all \( t \).

Definition: \( \mathbf{\ddot{c}}(t) = \mathbf{\dddot{c}}(t) = \) \( \mathbf{\dddot{c}}(t) \) is the acceleration of the curve.

\[ \Rightarrow \mathbf{\dddot{c}}(t) = (x''(t), y''(t), z''(t)) \]

Example: Suppose that the acceleration of a particle is \( \mathbf{\dddot{c}}(t) = -k \mathbf{\dot{c}}(t) \) (constant acc.).

Suppose that \( \mathbf{c}(0) = (0, 0, 1) \) and \( \mathbf{\dot{c}}(0) = (1, 1, 0) \).

Find the time \( t \) a spatial coordinate when the particle reaches \( z = 0 \).
\[ \text{Sol'n: want to find } t \text{ & } z(t) = (x(t), y(t), z(t)) \]

where \( z(t) = 0 \).

we know that \( \vec{a}(t) = (0, 0, -1) \) \( \forall t \)
\& \( \vec{a}''(t) = (x''(t), y''(t), z''(t)) \)

but \( \vec{v}(t) = (x'(t), y'(t), z'(t)) \)

So \( x'(t) = \text{const} \)
\( y'(t) = \text{const} \)
\( z'(t) = -t + \text{const} \)

But \( z'(0) = 1 \Rightarrow x'(t) = 1 \\
y'(0) = 1 \Rightarrow y'(t) = 1 \\
z'(0) = 0 \Rightarrow z'(t) = -t \]

Integrating again

\[ x(t) = t + \text{const} \text{ and } x(0) = 0 \Rightarrow x(t) = t \\
y(t) = t + \text{const} \text{ and } y(0) = 0 \Rightarrow y(t) = t \\
z(t) = -\frac{t^2}{2} + \text{const} \text{ & } z(0) = 1 \Rightarrow z(t) = 1 - \frac{t^2}{2} \]

We want \( t : z(t) = 0 \) so we solve
\[ -\frac{t^2}{2} = 0 \Rightarrow t = \sqrt{2} \quad (\text{bec } t > 0) \]

So \( \vec{z}(\sqrt{2}) = (\sqrt{2}, \sqrt{2}, 0) \) is the position of the particle when it crosses the \( z = 0 \) plane.

**Physics example** (How long is a planet’s year, knowing only its radius)

Suppose \( \vec{z}(t) = (r \cos \frac{st}{r}, r \sin \frac{st}{r}) \) is a circular orbit.

Then, the period is \( T = \frac{2\pi r}{s} \) \& \( \| \vec{z}(t) \| = r \) is the radius of the orbit.

The planet’s velocity is \( \vec{v}(t) = (-s \sin \frac{st}{r}, s \cos \frac{st}{r}) \).

And its speed is \( \| \vec{v}(t) \| = \sqrt{\left(-s \sin \frac{st}{r}\right)^2 + \left(s \cos \frac{st}{r}\right)^2} = s \).

The acceleration is \( \vec{a}(t) = \vec{v}'(t) = \left(-\frac{s^2}{r} \cos \frac{st}{r}, -\frac{s^2}{r} \sin \frac{st}{r}\right) \).

So \( \vec{a}(t) = -\frac{s^2}{r^2} (r \cos \frac{st}{r}, r \sin \frac{st}{r}) \).

(Or \( \frac{1}{2} \omega^2 \vec{z}(t) \) is the frequency, \( \omega^2 = \frac{\omega^2}{r^2} \), \( \omega \) is the frequency.)

So \( \vec{a}(t) = -\omega^2 \vec{z}(t) \).
But Physics tells us that force \( F = \frac{\text{mass}}{r^2} \)

and that \( \mathbf{F} = -\frac{G \text{ mass of planet \times mass of sun}}{r^3} \) is Newton's law of gravity.

So now \(-m \omega^2 \mathbf{z}(t) = -\frac{G \text{ mass of planet \times mass of sun}}{r^3} \)

\[ \text{taking } \omega \text{ on both sides } \]

\[ -m \frac{S^2}{r^2} \mathbf{z} = -\frac{G \text{ mass of planet \times mass of sun}}{r^3} \]

\[ \Rightarrow \]

\[ T^2 = \frac{r^3}{GM} \left(\frac{2\pi}{r}\right)^2 \]

Kepler's law.