(I) **Newton's Method**: Systems of non-linear equations

Want to solve $F(X) = 0$ where $X \in \mathbb{R}^n$

$F: \mathbb{R}^n \rightarrow \mathbb{R}^n$

_(system of n-equations, n unknowns)_

$(k+1)^{th}$ iteration

$$X^{(k+1)} = X^{(k)} + H^{(k)}$$

where

$$F'(X^{(k)}) H^{(k)} = -F(X^{(k)})$$

**Jacobian**: $F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$

**Example**: Perform two iterations of Newton's method on:

\[
\begin{align*}
2x^2 + 2xy + y^4 &= 3 \\
x^3y^5 - 2x^5y - x^2 &= -2
\end{align*}
\]

starting with $(1,1)$

**Solution**: $f_1(x,y) = 2x^2 + 2xy + y^4 - 3$

$f_2(x,y) = x^3y^5 - 2x^5y - x^2 = -2$
\[
F'(x, y) = \begin{bmatrix}
2y^2 + 2xy + 4x^2 & 2xy + x^2 \\
3x^2 y^2 - 10xy - 2x & 5x^2 y^4 - 2x^5
\end{bmatrix}
\]

\[
\Rightarrow F'(1,1) = \begin{bmatrix}
\frac{7}{9} & \frac{2}{3}
\end{bmatrix}
\]

\[
\Rightarrow x^{(1)} = x^{(0)} + h^{(0)} - F'(1,1)H^{(0)} = -F(1,1)
\]

Solve for \(H^{(0)}\), plug it in...

\[ \text{(II) Numerical Differentiation} \]

1. Using the Taylor series for \(f(x+h)\) & \(f(x-h)\)

\[
F'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f''''(\xi)
\]

error is \(O(h^4)\)

\[
F''(x) = \frac{1}{h^2} \left[f(x+h) - 2f(x) + f(x-h)\right] - \frac{h^2}{12} f''''(\xi)
\]

error is \(O(h^4)\)

You should know how to derive expressions like this.
Using Polynomial interpolation to approx. the deriv. at \( x_0 \)

\[
f(x) = \sum_{i=0}^{n} f(x_i) L_i(x) + \frac{1}{(n+1)!} f^{(n+1)}(x_c) \prod_{i=0}^{n} (x-x_i)
\]

Lagrange form of interpolating polynomial

\[
\Rightarrow \quad f'(x_0) = \sum_{i=0}^{n} f(x_i) L_i'(x_0) + \frac{1}{(n+1)!} f^{(n+1)}(x_c) \prod_{i=0}^{n} (x_0-x_i)
\]

Richardson Extrapolation: (for approx. \( f'(x) \))

1. Select \( h \), compute \( D(n,0) = \frac{f(h)}{2^n} \)
   where \( n = 0, \ldots, M \)
   \[ q(h) = \frac{f(x+h) - f(x-h)}{2h} \]

2. Compute
   \[
   D(n,k) = \frac{4^k}{4^k - 1} D(n,k-1) - \frac{1}{4^k - 1} D(n-1,k)
   \]
   where \( k = 1, \ldots, M \) & \( n = k, k+1, \ldots, M \)

Theorem: Suppose \( L = q(h) + \sum_{j=1}^{\infty} a_{ij} h^j \)

Then \( D(n,k-1) = L + \sum_{i=0}^{\infty} A_{ijk} (\frac{h^2}{2^n}) \approx O(h^2) \)
Numerical Integration

\[ F(x) \approx P(x) = \sum_{i=0}^{n} f(x_i) l_i(x) \]

Lagrange form of integrating polynomial

Integrate both sides

\[ \int_{a}^{b} f(x) \, dx \approx \sum_{i=0}^{n} A_i f(x_i) \quad \text{where} \quad A_i = \int_{a}^{b} l_i(x) \, dx \]

Examples: Trapezoid rule: \[ A_0 = A_1 = \frac{b-a}{2} \]

Simpson's rule: \[ x_0 = a, \ x_i = \frac{a+ib}{2}, \ x_2 = b \]

\[ A_0 = \frac{b-a}{6} = A_2, \ A_1 = \frac{4}{6}(b-a) \]

Method of undetermined coefficients to find the \( A_i \)'s

If you have \( n+1 \) nodes, define \( f_i(x) = x^i \quad i = 0, \ldots, n \)

Then \[ \int_{a}^{b} f_i(x) \, dx = \sum_{i=0}^{n} A_i f(x_i) \]

\[ \text{exactly} \quad \text{in equations with unknowns solve to get } A_i \text{'s} \]
Theorem: \[ \int_a^b f(x) \, dx - \sum_{i=0}^n A_i f(x_i) \leq \frac{M_n}{(n+1)!} \int_a^b \prod_{i=0}^n (x-x_i) \, dx \]
where \( M_n = \max_{x \in [a,b]} |f^{(n+1)}(x)| \)

when \( x_i = \cos \frac{(i+1)\pi}{n+2} \) (roots of Chebyshev poly of the 2nd kind)

then

\[ -1 \int_a^b f(x) \, dx - \sum_{i=0}^n A_i f(x_i) \mid \leq \frac{M_n}{(n+1)!} a^n \]