Recall: If $\vec{C}(t) = (x(t), y(t), z(t))$ is a path
then

Path Integral: $\int_{\vec{C}} F \cdot ds := \int_{a}^{b} F(\vec{C}(t)) \|\vec{C}'(t)\| \, dt$

Line Integral: $\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{a}^{b} \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t) \, dt$

New notation for the line integral:

$\vec{F} = (P, Q, R)$ & $d\vec{s} = (dx, dy, dz)$

We write $\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{\vec{C}} P \, dx + Q \, dy + R \, dz$

$= \int_{a}^{b} (P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt}) \, dt$

Warning: This is NOT the sum of 3 integrals. It is just another way to write $\int_{\vec{C}} \vec{F} \cdot d\vec{s}$.

We still need to parametrize and express everything in terms of the parameter.

Example: In the previous example

$\vec{C}(t) = (\cos t, \sin t, t), \quad 0 \leq t \leq 2\pi$

$\vec{F}(x, y, z) = x^2 \vec{i} + y^3 \vec{j} + z \vec{k}$

so $\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \int_{\vec{C}} x^2 \, dx + y^3 \, dy + z \, dz$
\[
= \int_0^1 \left( x^2(t) \frac{dx}{dt} + y^2(t) \frac{dy}{dt} + z(t) \frac{dz}{dt} \right) \, dt \\
= \int_0^1 \left( x \cos t \cdot (-\sin t) \, dt + y^2 \cdot \cos t \, dt + t \right) \, dt \\
= 0 + 0 + \frac{\pi^2}{2} \quad \text{(same as before)}
\]

**Example:**

Evaluate \( \int_C x^2 \, dx + xy \, dy \)

where \( \mathbf{c}(t) = (t, t^2), \quad 0 \leq t \leq 1 \)

**Solution:**

\[
\int_C x^2 \, dx + xy \, dy = \int_0^1 \left( x^2 \frac{dx}{dt} + xy \frac{dy}{dt} \right) \, dt \\
= \int_0^1 \left( t^2 \cdot 1 + t^3 \cdot 2t \right) \, dt \\
= \int_0^1 t^2 + 2t^4 \, dt = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}.
\]

**Remark:** Line integrals are independent of the parametrization as long as the parametrization is orientation preserving.
Let \( \vec{F} = x \, \vec{e}_1 + y \, \vec{e}_2 \)

\[
\int_{c} \vec{F} \cdot d\vec{s} = \int_{a}^{b} (t \, \vec{e}_1 + t^3 \, \vec{e}_2) \cdot \vec{c}'(t) \, dt
\]

\[
= \int_{0}^{1} \left( t \, \vec{e}_1 + t^3 \, \vec{e}_2 \right) \cdot \vec{c}'(t) \, dt
\]

\[
= \int_{0}^{1} \left( t \, \vec{e}_1 + t^3 \, \vec{e}_2 \right) \cdot \left( \vec{c}'(t) \right) \, dt
\]

---

**Line integral of gradient field**

Recall FTC: \( \int_{a}^{b} f'(t) \, dt = f(b) - f(a) \)

**Theorem:** (Fundamental Thm. of line integrals)

- \( F: \mathbb{R}^3 \to \mathbb{R} \), differentiable.
- \( C: [a, b] \to \mathbb{R}^3 \)

\[
\int_{C} (\nabla F) \cdot d\vec{s} = F(\vec{c}(b)) - F(\vec{c}(a))
\]

In words: If the field is a gradient field, only the end points matter.
Example: evaluate $\int_{\gamma} \nabla f \cdot ds$

where $f(x, y, z) = \cos z + \sin y - xyz$ and $\gamma$ is a trajectory that starts at $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ and ends at $(\frac{\pi}{2}, \frac{\pi}{4}, 1)$

Solution:

Let $\gamma(t), a \leq t \leq b$, be a path with $\gamma(a) = (\frac{\pi}{2}, \frac{\pi}{2}, 0) \& \gamma(b) = (\frac{\pi}{2}, \frac{\pi}{4}, 1)$

Then:

$$\int_{\gamma} \nabla f \cdot ds \overset{\text{FToLi}}{=} f(\gamma(b)) - f(\gamma(a))$$

$$= \cos \frac{\pi}{4} + \sin \frac{\pi}{2} - 0 - (\cos \frac{\pi}{4} + \sin 2\pi - \pi^2)$$

$$= 0 + 1 - (\frac{\sqrt{2}}{2} + 0 - \pi^2)$$

$$= 1 + 2\pi^2.$$
So far:
- Integrals of scalar fields along curves (path integral)
- Integrals of vector fields along curves (line integral)

Next:
Integrals over surfaces, but first we need learn to parametrize surfaces.

7.3 Parametrized Surfaces

E.g. the graph \( \mathcal{G} = \text{Function} \ F(x,y) \) is a surface.

Can have surfaces that are not the graph of a function.

E.g. torus (think surface of doughnut)

Definition: A parametrization of a surface is a 1-1 \( \mathcal{F} : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \). The surface is \( S = \mathcal{F}(D) \)

So \( \mathcal{F}(u,v) = (x(u,v), y(u,v), z(u,v)) \) with \( (u,v) \) in \( D \) (If \( x(u,v), y(u,v), z(u,v) \) are differentiable, we call \( S \) a smooth surface)
Parametrization of a plane:

Suppose that \( \alpha \) and \( \beta \) are \( \perp \) to \( P \) and that \((x_0,y_0,z_0) \in P \).
Then \( \vec{x} \) is not \( \perp \) to each other.

Then we can write any \((x,y,z) \in P\) as:
\[
(x,y,z) = (x_0,y_0,z_0) + u\vec{\alpha} + v\vec{\beta}
\]
for some \( u \) and \( v \in \mathbb{R} \).

So:
\[
\Phi(u,v) = x\alpha + y\beta + z
\]

Example: Find a parametrization of \( x + y + z = 1 \)

the point \((0,0,1)\) is on the plane.

The vectors \((1,1,0)\) and \((0,1,1)\) are \( \perp \) to the plane (why)
\[ \vec{D}(u,v) = (0,0,1) + (1,1,0)u + (0,1,-1)v \]

Check:
\[
\begin{align*}
  x(u,v) &= 0 + u \\
  y(u,v) &= 0 - u + v \\
  z(u,v) &= 1 - v
\end{align*}
\]
\[ x + y + z = 1 \]

Example: Find a parametrization of \( x + y + z = 1 \)

The point \((0,0,1)\) is on the plane.

The vectors \((1,1,0)\) & \((0,1,-1)\) are \( \perp \) to the plane (why)

\[ \vec{D}(u,v) = (0,0,1) + (1,1,0)u + (0,1,-1)v \]

Check:
\[
\begin{align*}
  x(u,v) &= 0 + u \\
  y(u,v) &= 0 - u + v \\
  z(u,v) &= 1 - v
\end{align*}
\]
\[ x + y + z = 1 \]

Tangent vectors to parametrized surfaces
Suppose that $\Phi(u,v)$ is diff. at $(u_0,v_0)$. Fix $u_0$ and look at the map $t \mapsto \Phi(u_0,t)$. (In other words, we have a map $\mathbb{R} \to \mathbb{R}^3$) now, which identifies a curve on the surface (the red one).

The vector tangent to this curve is given by

$$T_t = \frac{\partial \Phi}{\partial t} = \frac{\partial x}{\partial t} (u_0,v_0) \hat{i} + \frac{\partial y}{\partial t} (u_0,v_0) \hat{j} + \frac{\partial z}{\partial t} (u_0,v_0) \hat{k}$$

It is also tangent to the surface.

Similarly, $T_u = \frac{\partial \Phi}{\partial u} = \frac{\partial x}{\partial u} (u_0,v_0) \hat{i} + \frac{\partial y}{\partial u} (u_0,v_0) \hat{j} + \frac{\partial z}{\partial u} (u_0,v_0) \hat{k}$

$T_u$ and $T_t$ are both tangent to the surface.

So $T_u \times T_t$ is normal to it (provided $T_u \times T_t \neq 0$)

So, given $(u_0,v_0) \xrightarrow{\Phi} (x_0,y_0,z_0)$

to find the eq. of the tangent plane at $(x_0,y_0,z_0)$,

we calculate $N = T_u \times T_t = \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{array} \right| (u_0,v_0)$

Tang. Plane: $N \cdot (x-x_0,y-y_0,z-z_0) = 0$

$\Rightarrow N_1 (x-x_0) + N_2 (y-y_0) + N_3 (z-z_0) = 0$
Example:

Let \( \Phi : \mathbb{R}^2 \to \mathbb{R}^3 \) be given by

\[
\Phi(u,v) = (u \cos v, u \sin v, u^2 + v^2)
\]

Find the tangent plane at \( \Phi(1,0) \)

**Solution:**

\[
\begin{align*}
\frac{\partial}{\partial u} &= (\cos v, \sin v, 2u) \\
\frac{\partial}{\partial v} &= (-u \sin v, u \cos v, 2v)
\end{align*}
\]

\[
\nabla \Phi = \begin{bmatrix}
\cos v & \sin v & 2u \\
-u \sin v & u \cos v & 2v
\end{bmatrix}
\]

\[
P = \Phi(1,2) - \Phi(1,0) + \Phi(1,1) = -2\mathbf{i} + \mathbf{k}
\]

\[
(x_0, y_0, z_0) = (1 \cos 0, 1 \sin 0, 1^2 + 0^2) = (1, 0, 1)
\]

Thus, the tangent plane is:

\[
-2(x-1) + 0(y-0) + 1(z-1) = 0
\]

\[
-2x + z = -1
\]

**Note:** We say that a surface is **regular** or **smooth** at \( \Phi(u,v) \) if \( \frac{\partial}{\partial u} \times \frac{\partial}{\partial v} \neq 0 \) at \((u,v)\). We say that it is **regular** at all points \( \Phi(u,v) \in S \).

7.4 Area of a surface

**Note:** Just as we needed arclength to deal with path integrals, we will need surface area to compute surface integrals.
**Definition:** The surface area of a parametrized surface 

\[ A(S) = \iint_D \| \mathbf{T}_u \times \mathbf{T}_v \| \, du \, dv \]

where \( \Phi(D) \) is a regular, one-to-one differentiable elementary region.

Denoting \( \Phi(x,y) = \binom{\frac{\partial x}{\partial u}}{\frac{\partial x}{\partial v}} \) and \( \| \mathbf{T}_u \times \mathbf{T}_v \| \cdot du \cdot dv \), 

\[ \| \mathbf{T}_u \times \mathbf{T}_v \| = \sqrt{ \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial v} \right)^2 } \]

so \( A(S) = \iint_D \sqrt{ \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial v} \right)^2 } \cdot \Delta u \cdot \Delta v \)

To see why the definition of surface area makes sense:

\[ \Phi : u \times v \rightarrow \mathbb{R}^3 \]

so the area of the image of the small rectangle is 

\[ \| \Phi_u \times \Phi_v \| \cdot \Delta u \cdot \Delta v \]

Summing the areas of all such terms as \( \Delta u, \Delta v \rightarrow 0 \) gives \( A(S) \).

**Example:** Surface area of the cone \( z = \sqrt{x^2 + y^2} \)

where \( 0 \leq z \leq 1 \).

We parametrize \( \Phi(r, \theta) = (r \cos \theta, r \sin \theta, r) \), \( 0 \leq r \leq 1 \), \( 0 \leq \theta \leq 2\pi \).
So \( \overrightarrow{r} = \Phi = (\cos \theta, \sin \theta, 1) \)
\( \overrightarrow{0} = \Phi = (-r \sin \theta, r \cos \theta, 0) \)

\[ D \overrightarrow{r} \times \overrightarrow{0} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\cos \theta & \sin \theta & 0 \\
-r \sin \theta & r \cos \theta & 0
\end{vmatrix} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
1 & 0 & -\frac{r}{\sqrt{2}} \\
0 & 1 & \frac{r}{\sqrt{2}}
\end{vmatrix} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
-\frac{r}{\sqrt{2}} & \frac{r}{\sqrt{2}} & \frac{r}{\sqrt{2}}
\end{vmatrix} = \begin{vmatrix}
\overrightarrow{0} & \overrightarrow{r} \\
0 & r
\end{vmatrix} = r (\overrightarrow{0} \times \overrightarrow{r}) = (r \overrightarrow{0} \times \overrightarrow{r})
\]

\( \Rightarrow || \overrightarrow{r} \times \overrightarrow{0} || = \sqrt{(r \cos \theta)^2 + (-r \sin \theta)^2 + r^2} = \sqrt{r^2} = r \)

\( \Rightarrow A(\text{cone}) = \iiint_{\Phi} \sqrt{r} \, r \, dr \, d\theta = \sqrt{\frac{1}{2}} \cdot 2 \pi = \sqrt{\frac{1}{2}} \pi \)

Surface area of the graph of a function \( f(x, y) \)

we can parameterize \( S \) by \( (x, y, f(x, y)) \)

or \( (u, v, f(u, v)) \)

\[ \Rightarrow \overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
0 & \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\
1 & 0 & 1
\end{vmatrix} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
0 & \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\
0 & 1 & 1
\end{vmatrix} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & 1
\end{vmatrix}
\]

\( \Rightarrow A(S) = \iint_{D} \sqrt{(\frac{\partial f}{\partial u})^2 + (\frac{\partial f}{\partial v})^2 + 1} \, du \, dv \)

**Exercise**: Use this to compute the area of the cone.

\( z = \sqrt{x^2 + y^2} \)
Surfaces of revolution

Suppose $S$ is obtained by rotating $y = f(x)$, $a \leq x \leq b$ around the $x$-axis.

We can parameterize $S$ as $(u, f(u)\cos v, f(u)\sin v)$.

So

\[
\mathbf{T}_u = \left(1, f'(u)\cos v, f'(u)\sin v\right) \quad \Rightarrow \quad \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & f'(u)\cos v & f'(u)\sin v \\
0 & -f'(u)\sin v & f'(u)\cos v
\end{vmatrix}
\]

\[
\Rightarrow A(S) = \int_a^b \int_0^{2\pi} \sqrt{f'(u)^2 + (f'(u)\cos v)^2 + (f'(u)\sin v)^2} |f(u)|\,du\,dv
\]

\[
= \int_a^b \int_0^{2\pi} \sqrt{f'(u)^2 + 1} |f(u)|\,dv\,du = 2\pi \int_a^b \sqrt{f'(u)^2 + 1} |f(u)|\,du
\]

Note $A(S) = \int_2^\infty 2\pi |f(x)|\,dx$ where $c: [a, b] \to (t, f(t))$.

Exercise: use this to compute the area of a cone $z = \sqrt{x^2 + y^2}$. 