Chapter 8:

**Single variable calculus:**

- **Differential calculus**: e.g. rate of change, slope of tangent line
- **Integral calculus**: e.g. area under the graph of a function

**Fundamental theorem of calculus**

\[ \int_{a}^{b} f'(x) \,dx = f(b) - f(a) \]

**Remainder of the quarter:**
- Connecting vector integral calculus and vector differential calculus (tangent planes, Taylor's formula...)

\[ \Rightarrow \]

- Green's theorem
- Stoke's theorem
- Gauss's theorem

Green's theorem (8.1)

Relates two integral along a closed curve C in \( \mathbb{R}^2 \) to a double integral over the region enclosed by C.
Let's take a $y$-simple region $D$, with boundary $C$ and let $P: D \to \mathbb{R}$ be a function.
So we can write
\[
\iint_D \frac{\partial P}{\partial y}(x,y) \, dx \, dy = \iint_D \frac{\partial P}{\partial x}(x,y) \, dy \, dx
\]
\[
= \int_a^b P(x, \phi_2(x)) - P(x, \phi_1(x)) \, dx
\]
\[
\text{(I) by the FTC}
\]
\[
\text{but } (x, \phi_2(x)) \text{ is the parametrization of the top part of } C \text{ going from } a \text{ to } b
\]
\[
\text{and } (x, \phi_1(x)) \text{, } a \leq x \leq b \text{ is the parametrization of the bottom part of } C \text{ going from } a \text{ to } b
\]
\[
\text{So } \int_a^b P(x, \phi_2(x)) \, dx = \int_{C_{\text{top}, a \rightarrow b}} P(x, y) \, dx \quad \text{(II)}
\]
\[
\text{and } \int_a^b P(x, \phi_1(x)) \, dx = \int_{C_{\text{bottom}, a \rightarrow b}} P(x, y) \, dx \quad \text{(III)}
\]
\[
= - \int_{C_{\text{bottom}, b \rightarrow a}} P(x, y) \, dx
\]
\[
\text{Also } \int_{C_{\text{left}}} P(x, y) \, dx = \int_{C_{\text{right}}} P(x, y) \, dx = 0 \quad \text{(IV)}
\]
\[
\text{because } x \text{ is constant}
\]
So:
\[ \iint_D \frac{\partial Q}{\partial y} \, dx \, dy = \int_{C^+} P \, dx = -\int_{C^-} P \, dx \]

Substituting (I), (II) & (III) in (*) for a y-simple region D

Similarly:
\[ \iint_D \frac{\partial Q}{\partial x} \, dx \, dy = \int_{C^+} Q \, dy = -\int_{C^-} Q \, dy \]

For an x-simple region D

\[ \text{Green's Theorem: Let } D \text{ be a simple region and let } C \text{ be its boundary. Let } P : D \cap \mathbb{R}^2 \to \mathbb{R} \text{ and } Q : D \cap \mathbb{R}^2 \to \mathbb{R} \text{ have continuous partials.} \]
\[ \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \int_{C^+} P \, dx + Q \, dy \]

Counter clockwise
If the region is not simple, we can break it up into simple regions and sum them up.

Note: As you traverse the boundary, the region should be to your left.

Example: Evaluate using Green’s Theorem

\[
\oint_C y^3 \, dx - x^3 \, dy = 0
\]

\[
\iint_D (P_y - Q_x) \, dx \, dy = \iint_{\text{Unit Disk}} -3x^2 - 3y^2 \, dx \, dy
\]

\[
= \int_0^{2\pi} \int_0^1 -3r^2 \, r \, dr \, d\theta
= \int_0^{2\pi} \left[ -\frac{3}{4} r^4 \right]_0^1 \, d\theta
= -\frac{3\pi}{4}
\]

Verify this by evaluating \( \oint_C y^3 \, dx - x^3 \, dy \) directly.
\[ \int_{\mathbb{R}} y^3 \, dx - x^3 \, dy = \int_{0}^{2\pi} ((\sin \theta)^3 - (\cos \theta)^3) \cdot (\cos \theta \sin \theta) \, d\theta \]
\[= \int_{0}^{2\pi} \sin^4 \theta - \cos^5 \theta \, d\theta = \ldots = -6 \frac{\pi}{4} \]

**Trig identities**

**Important application of Green's Theorem (Area of a region)**

\[ \iint_{D} \, dx \, dy = A = \frac{1}{2} \oint_{\partial D} x \, dy - y \, dx \]

**Boundary of** \( D \) **oriented counterclockwise**

**Proof:**

\[ \iint_{D} \, dx \, dy = \frac{1}{2} \iint_{D} (1 + 1) \, dx \, dy = \frac{1}{2} \iint_{D} \frac{\partial}{\partial x} (\frac{\partial}{\partial y}) \, dx \, dy \]
\[= \frac{1}{2} \oint_{\partial D} x \, dy - y \, dx \]

**Green's theorem**

**Example:**

**Compute the area of the hyperboloid** \( S: \{ (a \cos^3 \theta, a \sin^3 \theta), 0 \leq \theta \leq 2\pi \} \)

\[ (x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}) \]

\[ C(\theta) \]

\[ C'(\theta) \]

\[ A = \iint_{D} -y \, dx + x \, dy = \frac{1}{2} \int_{0}^{2\pi} (-a \sin^2 \theta, a \cos^2 \theta) \cdot (3a \sin^2 \theta \cos \theta, 3a \sin^2 \theta \cos \theta) \, d\theta \]
\[
\frac{1}{2} \int_0^{2\pi} \left( 3a^2 \sin^2 \theta \cos^2 \theta + 3a^2 \sin^2 \theta \cos^2 \theta \right) d\theta = \frac{3}{2} a^2 \int_0^{\pi} \left( \sin^2 \theta \cos^2 \theta \right) d\theta
\]

\[
\text{because} \quad (\sin \theta)^2 = \frac{1 - \cos^2 \theta}{2} \quad \Rightarrow \quad \frac{1}{4} \cdot \frac{(\sin \theta)^2}{2} = \frac{1 - \cos^2 \theta}{2}
\]

\[
= \frac{3}{8} a^2 \int_0^{\pi} \frac{1 - \cos^2 \theta}{2} d\theta = \frac{3}{8} \pi a^2
\]

**Curl of a vector field**

\[
\text{Del operator} \quad \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}
\]

\[
\Rightarrow \quad \text{gradient} \quad \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}.
\]

\[
\text{curl} \quad \nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix}
\]

\[
= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}
\]

**Example:** Compute the curl of \( \vec{F} = y \hat{i} - x \hat{j} \)

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & 0
\end{vmatrix}
\]

\[
= \left( \frac{\partial}{\partial y} \cdot 0 - \frac{\partial}{\partial z} \cdot (-x) \right) \hat{i} + \left( \frac{\partial}{\partial z} \cdot y - \frac{\partial}{\partial x} \cdot 0 \right) \hat{j} + \left( \frac{\partial}{\partial x} \cdot (-x) - \frac{\partial}{\partial y} \cdot y \right) \hat{k}
\]

\[
= 0 \hat{i} + 0 \hat{j} + (-2) \hat{k}
\]
Curl is associated with rotations. So, informally, if you "drop" a small object with sides $dx, dy, dz$ into a vector field $\vec{F}$, it would rotate about its axis with angular velocity $\frac{1}{2}(\nabla \times \vec{F})$. See book/more on this later.

A vector field with $\nabla \times \vec{F} = 0$ is called irrotational.

**Vector form of Green’s theorem**

Suppose $\vec{F} = P(x,y)\hat{x} + Q(x,y)\hat{y}$ and suppose we have a region $D$ with boundary $\partial D$ (positively oriented).

Then $\int_{\partial D} P\,dx + Q\,dy = \int_{D} \vec{F} \cdot d\vec{s}$

Moreover, $\iint_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \,dx\,dy = \iint_{D} (\nabla \times \vec{F}) \cdot \hat{k} \,dx\,dy$

Let $\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = 0 \hat{x} + 0 \hat{y} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{z}$

which implies $(\nabla \times \vec{F}) \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

So: $\iint_{D} \vec{F} \cdot d\vec{s} = \iint_{D} (\nabla \times \vec{F}) \cdot \hat{k} \,dx\,dy$
Example: Let $\vec{F} = (xy, x-y)$

Compute $\iint_D \nabla \times \vec{F} \cdot \vec{n} \, dA$ where $D$ is the region in the first quadrant bounded by the curves $y = x^2$ and $y = x$.

By Green's theorem, $\iint_D \nabla \times \vec{F} \cdot \vec{n} \, dA = \oint_{\partial D} \vec{F} \cdot d\vec{s}$

First: along the parabola from $x = 0$ to $x = 1$ (parametrized as $\vec{r}(x) = (x, x^2)$)

\[\int_0^1 (F_1, F_2) \cdot (1, 2x) \, dx = \int_0^1 (x^2, x^3) \cdot (1, 2x) \, dx = \int_0^1 x + 2x^3 \, dx = \frac{1}{2} + \frac{2}{4} = \frac{1}{2} \]

Second: along the line $y = x$ from $x = 1$ to $x = 0$

\[\int_1^0 (F_1, F_2) \cdot (1, 1) \, dx = -\int_1^0 2x \, dx = -x^2 \big|_1^0 = -\frac{1}{2} \]

\[\Rightarrow \oint_{\partial D} \vec{F} \cdot d\vec{s} = 1 - \frac{1}{2} = \frac{1}{2} = \iint_D \nabla \times \vec{F} \cdot \vec{n} \, dA\]
Stoke's theorem \((8.2)\)

Green's theorem dealt with plane regions. Stoke's theorem relates the line integral of a vector field around a simple closed curve \(C\) to an integral over a surface \(S\) with \(C\) as its boundary.

\[ \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \]

**IF** \(C\) is a closed curve in space, and \(S\) is an surface bounded by \(C\), then

Rule to determine orientation: Walking along \(C\) in the +ve direction, with \(S\) do your left, \(\mathbf{n}\) should be pointing up.

**Observation:** It doesn't matter what \(S\) is as long as its boundary is \(C\).
Example: Verify Stokes' theorem for

\[ \vec{F} = z \vec{z} + x \vec{\hat{x}} + y \vec{\hat{y}} \]

C = unit circle in the xy-plane oriented ccw.

S: \( z = 1 - x^2 - y^2 \)

\( \vec{F} \cdot d\vec{s} = \int_{C} \vec{F} \cdot \vec{C}(\theta) \, d\theta = \int_{0}^{2\pi} \vec{F} \cdot ((\cos \theta, \sin \theta, 0) \times (-\sin \theta, \cos \theta, 0)) \, d\theta \\
= \int_{0}^{2\pi} \cos \theta \, d\theta = \int_{0}^{2\pi} \cos^{2} \theta \, d\theta = \pi. \)

On the other hand,

\[ \int_{S} (\nabla \times \vec{F}) \cdot d\vec{s} = ? \]

\( \nabla \times \vec{F} = \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{array} \right| = x + \hat{z} + \hat{x} \)

Since S: \((x, y, 1 - x^2 - y^2)\)
then \( \nabla \times \vec{F} = \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, 1 \right) = (2x, 2y, 1) \)

\[ \int_{0}^{2\pi} \int_{0}^{1} \frac{r}{2} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} = \frac{1}{2} \]
\[ \text{So } \int_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\text{unit disk}} 2x + 2y + 1 \, dx \, dy \]

\[ = \int_0^{2\pi} \int_0^1 (2r \cos \theta + 2r \sin \theta + 1) \, r \, dr \, d\theta \]

\[ = \int_0^{2\pi} \left( \int_0^1 (2r^2 \cos \theta + 2r^2 \sin \theta + r) \, dr \right) \, d\theta \]

\[ = \int_0^{2\pi} \left( \frac{2}{3} \cos \theta + \frac{1}{2} \sin \theta + \frac{1}{2} \right) \, d\theta \]

\[ = \left[ \frac{2}{3} \sin \theta - \frac{1}{2} \cos \theta + \frac{1}{2} \theta \right]_0^{2\pi} = 0 \]

**Example:** Let \( S \) be a surface whose boundary is the circle \( x^2 + y^2 = 1 \), where \( S \) lies above the \( xy \)-plane with normal vector having the \( k \) component. Let \( \vec{F} = y \hat{i} - x \hat{j} + e^{x^2} \hat{k} \) and compute \( \int_S (\nabla \times \vec{F}) \cdot d\vec{S} \)

**Solve:** By Stoke’s theorem, \( \int_S (\nabla \times \vec{F}) \cdot d\vec{S} \)

\[ = \int_C \vec{F} \cdot d\vec{S} \] where \( C : (\cos \theta, \sin \theta, 0) \) and \( 0 \leq \theta \leq 2\pi \) (hence oriented ccw)

\[ = \int_0^{2\pi} \left( \sin \theta \hat{i} - \cos \theta \hat{j} \right) \cdot \left( -\sin \theta \hat{i} + \cos \theta \hat{j} + 0 \hat{k} \right) \, d\theta \]

\[ = \int_0^{2\pi} \left( -\sin^2 \theta + \cos^2 \theta \right) \, d\theta \]

\[ = \int_0^{2\pi} \left( 1 - 2\sin^2 \theta \right) \, d\theta \]

\[ = \int_0^{2\pi} (1 - 1 + \cos 2\theta) \, d\theta \]

\[ = \int_0^{2\pi} \cos 2\theta \, d\theta \]

\[ = \left[ \frac{1}{2} \sin 2\theta \right]_0^{2\pi} = 0 \]

**Note:** we didn’t care what \( S \) was explicitly.
**Important Remark:** For a surface \( S \) with no boundary, \( \int_S (\nabla \times F) \cdot d\mathbf{s} = 0 \) e.g., the sphere.

By Stoke's theorem.

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**Gauss's theorem (8.4)**

But first, divergence of a vector field.

If \( \mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k} \), then

\[
\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}
\]

(dot product of \( \nabla \) with \( \mathbf{F} \))

Note: divergence of \( \mathbf{F} \) is a scalar.

**Example:** Compute the divergence of

\[
\mathbf{F} = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k}
\]

\[
\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial y} (xy) + \frac{\partial}{\partial z} (z) = 2x + xz + 0
\]
Physical interpretation: If \( \mathbf{F} \) is the velocity field of a fluid, then \( \nabla \cdot \mathbf{F} \) is the rate of expansion per unit volume under the flow of the gas.

**Theorem:** \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \)

**Exercise:** Check this using the definitions.

**Gauss's Theorem:** If \( S \) is a closed surface bounding a region \( W \), with normal pointing outward, and if \( \mathbf{F} \) is a vector field defined over \( W \) and differentiable:

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} \, dV
\]

**Example:** \( \mathbf{F} = 2x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \)

\( S \): unit sphere. Evaluate \( \iiint_W \mathbf{F} \cdot d\mathbf{S} \).

\[
\iiint_W \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} \, dV = \iiint_W (2 + 2y + 2z) \, dV
\]

Evaluating over the sphere, we get:

\[
= 2 \iiint_{ball} dV + 2 \iiint_{ball} y \, dV + 2 \iiint_{ball} z \, dV = \frac{8}{3} \pi (1)^2
\]
**Interpretation**

At a point \((x, y, z)\), \((\nabla \mathbf{F})\) \(\left|_{(x_0, y_0, z_0)}\right\) is the rate of outward flow per unit volume (or expansion).

So, in a sense, Gauss's theorem tells us that the total outward flow \(\iiint_{W} (\nabla \mathbf{F}) \cdot dV\) equals the total flux out of the boundary.

**Example** Compute \(\iiint_{S} \mathbf{F} \cdot dS\) where

\(S\) is the surface of the box: \(0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\)

and \(\mathbf{F} = \left(3x + e^{xyz}\right) \mathbf{i} + \left(\frac{z}{y} + 5 \sinh(z+x)\right) \mathbf{j} + (x^2 + y^2) \mathbf{k}\)

**Solution** by Gauss's theorem \(\iiint_{V} \nabla \mathbf{F} \cdot dV = \iiint_{W} \nabla \cdot \mathbf{F} \cdot dV\)

\[\iiint_{V} \nabla \mathbf{F} \cdot dV = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \left(\frac{3}{x} + e^{xyz}\right) + \left(\frac{1}{y} + 5 \cosh(z+x)\right) + \left(x^2 + y^2\right) dV\]

What made this a Gauss's theorem problem?
- Field was terrible and \(\nabla \mathbf{F}\) was nice.
Applications (Physics)

Given a charge density \( \rho(x, y, z) \) in a region \( W \), the field \( \vec{E} \) satisfies \( \nabla \cdot \vec{E} = \rho \) (given)

\[ \Rightarrow \iiint_W \nabla \cdot \vec{E} \, dV = \iint_S \vec{E} \cdot d\vec{S} \]

- total charge \( Q \)
- Flux out of the surface inside \( W \)

Remark: A two-dimensional version of Gauss' divergence theorem, i.e., with \( \vec{F} = P \hat{i} + Q \hat{j} \) and region \( D \) bounded by closed curve \( \partial D \) is

\[ \iint_D (\nabla \cdot \vec{F}) \, dx \, dy = \oint_{\partial D} (\vec{F} \cdot \hat{n}) \, ds \]

In fact, this is the divergence form of Green's theorem
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Conservative Vector Fields (8.3)

\[ \oint_C \nabla f \cdot ds = f(\mathbf{z}(b)) - f(\mathbf{z}(a)) \]

If the field \( \mathbf{F} \) is a gradient vector field i.e. \( \mathbf{F} = \nabla f \) for some function \( f(x,y,z) \), then the line integral is path independent.

Example:
\[ \mathbf{F} = (\cos(x) \cos(y) + x^2 \cos(z), -\sin(x) \sin(y) + x \cos(z), x^2 \sin(z)) \]
\[ \mathbf{c} = (\cos(\pi t) \sin(\pi t), t^2, \pi t) \], \( 0 \leq t \leq 1 \)

Evaluate \( \int_C \mathbf{F} \cdot ds \)

Solution: \( \mathbf{F} = \nabla f \) where \( f = \sin(x) \cos(y) + e^{2z} \)
\[ \Rightarrow \int_C \mathbf{F} \cdot ds = f(\mathbf{z}(b)) - f(\mathbf{c}(a)) \]
\[ = f(\cos(2\pi t) \sin(2\pi t), t^2, \pi t) - f(\cos(2\pi/6) \sin(2\pi/6), 0^2, 0) \]
\[ = f(0,1) - f(0,0,0) = \sin(0) \cos(1) + e^{2\pi} - \sin(0) \cos(0) - e^{0} = 0 \]

(much easier than \( \int_0^1 f \circ c'(t) \, dt \)).

We'd like to know when vector fields are gradients.
Theorem: Let \( \overrightarrow{F} \) have continuous partial derivatives.

All these statements are equivalent:

(i) \( \int_C \overrightarrow{F} \cdot d\overrightarrow{s} = 0 \) for all oriented simple closed curves.

(ii) \( \int_C \overrightarrow{F} \cdot d\overrightarrow{s} = \int_C \overrightarrow{F} \cdot d\overrightarrow{s} \) for simple oriented curves with the same endpoints.

(iii) \( \overrightarrow{F} = \nabla f \) for some \( f \).

(iv) \( \nabla \times \overrightarrow{F} = 0 \)

We call such a field \( \overrightarrow{F} \) conservative.

In the above example, how did we know that \( \overrightarrow{F} \) was conservative? How did we find \( f \)?

Answer: either by inspection

\[
\begin{align*}
\nabla \times \overrightarrow{F} &= \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x^2 & y^2 & z^2
\end{vmatrix} \\
&= (2x^2z^2 + y^2z^2 - (x^2y^2 + x^2z^2 + y^2z^2))
\end{align*}
\]

\( = 0 \)
If such an \( F \) exists:

\[
\begin{align*}
\frac{\partial F}{\partial x} &= \cos x \cos y + y^2 e^{x^2} \\
\frac{\partial F}{\partial y} &= -\sin x \sin y + x^2 e^{x^2} \\
\frac{\partial F}{\partial z} &= -2xy e^{x^2}
\end{align*}
\]

Thus \( \frac{\partial F}{\partial z} \neq -\frac{\partial F}{\partial y} \).

\[
\frac{\partial F}{\partial z} = x^2 e^{x^2} = f(x,y,z) = e^{x^2}g(x,y) + h(x)
\]

So \( \frac{\partial F}{\partial y} = -\sin(x)\sin(y) \Rightarrow g(x,y) = \sin(x)\cos(y) + k(x) \).

So we have \( f(x,y,z) = e^{x^2} + \sin(x)\cos(y) + k(x) \).

\[
\frac{\partial F}{\partial x} = e^{x^2}x^2e^{x^2} + \cos x \cos y \sin y + k'(x)
\]

So \( F'(x) = 0 \) works.

**Remark:** In 2 dimensions (i.e., the plane case),

\[
\nabla F = (\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} \hat{\mathbf{z}}) \hat{\mathbf{r}}
\]

So \( F = P \hat{\mathbf{r}} + Q \hat{\mathbf{z}} \) is conservative when \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \). (So there is an \( F : \nabla g = \hat{\mathbf{r}} \)).

**Example:**

\[
F = (2xy - \sin x) \hat{\mathbf{r}} + x^2 \hat{\mathbf{z}}
\]

is conservative because

\[
\begin{align*}
\frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} = 2x \\
\frac{\partial P}{\partial x} &= \sin x
\end{align*}
\]

To find \( F \), solve \( \frac{\partial F}{\partial x} = 2xy \sin x, \quad \frac{\partial F}{\partial y} = x^2 \)

\[
\Rightarrow F(x,y) = x^2 y + \cos x
\]
Example

Let \( \vec{C} : [0, 1] \rightarrow \mathbb{R}^2 \) be given by \((e^{2t}, \sin(\pi t))\)

compute \( \int_C 2x \cos y \, dx - x^3 \sin y \, dy \)

This means they want \( \int_C \vec{F} \cdot d\vec{s} \) where \( \vec{F} = (2x \cos y, -x^3 \sin y) \)

\( \frac{\partial}{\partial y} (2x \cos y) = -2x \sin y \neq \frac{\partial}{\partial x} (-x^3 \sin y) \)

So \( \vec{F} \) is conservative & \( \int_C \vec{F} \cdot d\vec{s} = f(\vec{C}(1)) - f(\vec{C}(0)) \)

= \( f(e, 1) - f(1, 0) \)

Moreover \( f(x, y) = x^2 \cos y \) so \( \int_C \vec{F} \cdot d\vec{s} = e^2 \cos(1) - 1 \)

Recall: \( \nabla \times (\nabla f) = 0 \) & \( \nabla \cdot \vec{F} = 0 \Rightarrow \vec{F} = \nabla f \)

It is also true that \( \nabla \cdot (\nabla \times \vec{G}) = 0 \) & \( \nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla \times \vec{G} \)

has conservative potentials.

Maxwell's Equations

- Physically derived laws.
- Relate \( \vec{E}(x, y, z, t) \) & \( \vec{H}(x, y, z, t) \) the electric & magnetic fields (time varying) to each other and to \( \rho(x, y, z, t) \), the charge density, and \( \vec{J}(x, y, z, t) \), the current density.
The equations

\[ \nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad \text{(Gauss's Law)} \]  
\[ \nabla \cdot \vec{H} = 0 \quad \text{(Faraday's law)} \]  
\[ \nabla \times \vec{E} = -\partial \vec{H} / \partial t \quad \text{(Amperes law)} \]  
\[ \nabla \times \vec{H} = \mu_0 (\varepsilon_0 \partial \vec{E} / \partial t + \vec{J}) \quad \text{(Faraday's law)} \]

\[ \text{Permeability} \quad \text{Permittivity} \]

* For example 1 tells us that \( \int \int \int \rho dV = \text{total charge} = \int \int \int \vec{E} \cdot d\vec{s} \)

* 2 has no RHS because there are no free magnetic charges.

* 3 & 4 tell us that a time varying Electric field induces a magnetic field \& vice versa.

Let’s look at the equations in vacuum i.e. \( \vec{J} = 0, \rho = 0 \)

\[ \nabla \times \vec{E} = -\partial \vec{H} / \partial t \]

\[ \nabla \times \vec{H} = \mu_0 \varepsilon_0 \partial \vec{E} / \partial t \]

So

\[ \nabla \times (\nabla \times \vec{E}) = -\partial / \partial t \left( \mu_0 \varepsilon_0 \partial \vec{E} / \partial t \right) = -\mu_0 \varepsilon_0 \partial^2 \vec{E} / \partial t^2 \]

\[ \text{(Vector Laplacian)} \]

But \( \nabla \times (\nabla \times \vec{E}) = \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} \) (see section 4.4)

So

\[ \nabla \left( \nabla \cdot \vec{E} \right) - (\partial^2 \vec{E} / \partial x^2 + \partial^2 \vec{E} / \partial y^2 + \partial^2 \vec{E} / \partial z^2) = -\mu_0 \varepsilon_0 \partial^2 \vec{E} / \partial t^2 \]

\[ \Rightarrow \mu_0 \varepsilon_0 \partial^2 \vec{E} / \partial t^2 = \nabla^2 \vec{E} \]

Similarly

\[ \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2} = \nabla^2 \vec{H} \]

So EM waves propagate at the speed of light

\[ \text{(Maxwell suggested)} \]

and light is an EM wave!