Problem 7.1

(iii) $\mathbb{Z}^+$, because every positive integer is greater than or equal to itself.

(iv) $\emptyset$, because no positive integer is greater than or equal to every positive integer. In particular, $n$ is not greater than or equal to $n + 1$.

Problem 7.2

(iii) This is true. Let $m \in \mathbb{Z}^+$, and set $n = m$. Since $m \leq m$, the statement is satisfied.

(iv) True. Let $m = 1$. Then for all $n \in \mathbb{Z}^+$ we have $m \leq n$.

(v) True. Let $n \in \mathbb{Z}^+$ and set $m = n$. Since $n \leq n$, the statement is satisfied.

Problem 7.4

(vi) False. Suppose there did exist such a $y$. When $x = 0$, we have $xy = 0 \neq 1$.

(vii) True. Let $n \in \mathbb{Z}^+$. If $n$ is even, then $n$ is even or odd. If $n$ is not even, then by definition $n$ is odd, so it is even or odd. In either case the statement holds.

Problem 7.5

Let $n \in \mathbb{Z}$ and suppose there exists $q \in \mathbb{Z}$ such that $n = 2q + 1$. Then we have

$$n^2 = (2q + 1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1.$$ 

Since $2q^2 + 2q$ is an integer, we can set $p = 2q^2 + 2q$. Thus, there exists $p \in \mathbb{Z}$ such that $n^2 = 2p + 1$ as desired.

Problem 7.7

Let $A, B, C$, and $D$ be sets. Let $(x, y) \in (A \times B) \cup (C \times D)$. Then $(x, y) \in A \times B$ or $(x, y) \in C \times D$.

First, suppose $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$, so we have $x \in A \cup C$ and $y \in B \cup D$. Therefore $(x, y) \in (A \cup C) \times (B \cup D)$.

Now suppose $(x, y) \in C \times D$. Then $x \in C$ and $y \in D$, so we have $x \in A \cup C$ and $y \in B \cup D$. Therefore $(x, y) \in (A \cup C) \times (B \cup D)$.

To see that these sets are not always equal, let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$. Then we have

$$A \times B = \{(1, 2)\} \text{ and } C \times D = \{(3, 4)\}$$
so that

\[(A \times B) \cup (C \times D) = \{(1, 2), (3, 4)\}.
\]

However,

\[(A \cup C) \times (B \cup D) = \{(1, 2), (1, 4), (3, 2), (3, 4)\}.
\]

**Problem 10**

(i) First we’ll show \((a, \infty) \subseteq [b, \infty) \implies a \geq b\). Let \((a, \infty) \subseteq [b, \infty)\) and suppose toward a contradiction that \(a < b\). Let \(x = a + \frac{b-a}{2}\). Then \(x > a\), so \(x \in (a, b)\). Since \((a, \infty) \subseteq [b, \infty)\), this implies \(x \in [b, \infty)\), so \(x \geq b\). On the other hand,

\[
x = a + \frac{b-a}{2} = \frac{a}{2} + \frac{b}{2} < \frac{b}{2} + \frac{b}{2} \text{ since } a < b
\]

Thus \(x < b\), which contradicts the fact that \(x \geq b\). Therefore we may conclude that \(a \geq b\).

Now we’ll show \(a \geq b \implies (a, \infty) \subseteq [b, \infty)\). Let \(x \in (a, \infty)\). Then \(x > a \geq b\), so \(x \geq b\). Thus \(x \in [b, \infty)\), so we may conclude that \((a, \infty) \subseteq [b, \infty)\).

(ii) First we’ll show \([a, \infty) \subseteq (b, \infty) \implies a > b\). Let \([a, \infty) \subseteq (b, \infty)\), and suppose toward a contradiction that \(a \leq b\). Let \(x = b\). Then \(x \geq a\), so \(x \in [a, \infty)\). Since \([a, \infty) \subseteq (b, \infty)\), this implies \(x \in (b, \infty)\), which means \(x > b\). However \(x = b\), so this is impossible. Therefore \(a > b\).

Now we’ll show \(a > b \implies [a, \infty) \subseteq (b, \infty)\). Let \(x \in [a, \infty)\). By definition, \(x \geq a > b\), so we have \(x > b\). Thus, \(x \in (b, \infty)\). Therefore \([a, \infty) \subseteq (b, \infty)\).

**Problem 12**

The statement “There is a greatest number in the set A” can be expressed in quantifiers as

\[\exists x \in A, \forall y \in A, x \geq y.\]

The negation of the statement is “There does not exist a greatest number in the set A.” We can express this in quantifiers as

\[\forall x \in A, \exists y \in A, x < y.\]