Chapter 10: Counting

Cardinality of finite sets

When we count the elements of a finite set, we are effectively constructing a bijection from the set

\[ \mathbb{N}_n := \{1, 2, \ldots, n\} = \{i \in \mathbb{Z} \mid 1 \leq i \leq n\} \]

to the set being counted.

**Definition**

Given a set \( X \), if there is a bijection \( f: \mathbb{N}_n \rightarrow X \), then we say that the cardinality of \( X \) (the number of elements in \( X \)) is \( n \). We write

\[ |X| = n. \]

**Example:** Let \( X = \{7, 8, 9, 10, 11\} \).

Then \( |X| = 5 \) because \( \exists f: \mathbb{N}_5 \rightarrow X \), where \( f \) is a bijection. In particular, we may choose

\[ f(1) = 7, f(2) = 8, f(3) = 9, f(4) = 10, f(5) = 11. \]
Of course, there are many other bijections between \( N_5 \) and \( X \). Are there bijections between \( N_m \) and \( X \) with \( m \neq n \)? If there are, we're in trouble because a set would have two cardinalities.

**Proposition:** Suppose \( f : N_m \to X \) and \( g : N_n \to X \) are both bijections. Then \( m = n \).

**Strategy:** Since \( g \) is a bijection, \( g^{-1} \) is a bijection (\( g^{-1} : X \to N_n \)).

So \( g^{-1} \circ f : N_m \to N_n \) is a bijection. (Why?)

So we would be done if we can prove that 
(having a bijection \( h : N_m \to N_n \)) \( \Rightarrow (m = n) \). \( \text{(*)} \)

We can prove (*) by showing:

(h : \( N_m \to N_n \) is an injection) \( \Rightarrow (m \leq n) \). \( \text{(**)} \)
In particular, to prove (*) using (**), we observe that:

\[ h \text{ is a bijection} \Rightarrow h \text{ is an injection} \Rightarrow m \leq n \]

\[ h \text{ is a bijection} \Rightarrow h^{-1} \text{ is a bijection} \Rightarrow h^{-1} \text{ is an injection} \]

\[ \text{with } h^{-1} : N^n \rightarrow N^m \]

\[ \Rightarrow n \leq m \]

We would then have \((n \leq m) \land (m \leq n) \Rightarrow (m = n)\) so we'd be done. So now the whole proof would work provided (***) holds.

**Lemma:** If there is an injection \( N^m \rightarrow N^n \) then \( m \leq n \)

*Note: The Pigeonhole Principle*

We will prove this lemma shortly. For now, let us assume it is true, and prove the Proposition.
**Proof of proposition:** Since \( f : N_m \to X \) and \( g : N_n \to X \) are injections, they are invertible and their inverses \( f^{-1} : X \to N_m \) \& \( g^{-1} : X \to N_n \) are also bijections.

Thus \( g^{-1} \circ f : N_m \to N_n \) is a bijection, hence an injection. By the lemma above, we now have \( m \leq n \).

Similarly \( f^{-1} \circ g : N_n \to N_m \) is a bijection, hence an injection. By the lemma above, we now have \( n \leq m \).

Together, \( n \leq m \) \& \( m \leq n \) yield \( m = n \).

Now, let us go back and prove the Pigeonhole Principle.

**Proof of Pigeonhole Principle**

We'd like to prove that if \( f : N_m \to N_n \) is injective then \( m \leq n \). We will do this by induction on \( n \).
Base case \((n=1)\): Consider \(f: N_m \rightarrow N_1\) injective and note that we must then have \(f(i) = 1 \quad \forall i \in N_m\) but injectivity requires \(f(i) = f(j) \Rightarrow i = j\), so there is only one element in \(N_m\), so \(m = 1\). We have in the case \(m \leq 1\).

Inductive step: Here we assume that

\[\forall m \left( \text{if there is an injection } f: N_m \rightarrow N_k \text{, then } m \leq k \right)\]

\[P(k)\]

and we'd like to prove

\[\forall m \left( \text{if there is an injection } f: N_m \rightarrow N_{k+1} \text{, then } m \leq k+1 \right)\]

\[P(k+1)\]

So we start by assuming that \(f: N_m \rightarrow N_{k+1}\) is an injection.

We consider two cases:

1. \(f(i) \leq k \quad \forall i \in N_m\)

2. \(\exists i_0 \in N_m \text{ s.t. } f(i) = k+1\).

These cases are mutually exclusive, and exhaustive, so if we prove it in each case, we're done.
Case 1: If \( f(i) \leq k \) \( \forall i \in \mathbb{N}_m \), we can define
\[ f_1 : \mathbb{N}_m \to \mathbb{N}_k \] via \( f_1(i) = f(i) \) \( \forall i \in \mathbb{N}_m \).

Since \( f \) is injective, \( f_1 \) is also injective (why?)
So by the inductive hypothesis \( m \leq k < k + 1 \)
So we are done.

Case 2: Suppose \( f(i_0) = k + 1 \) for some \( i_0 \in \mathbb{N}_m \).

Define a new function \( g : \mathbb{N}_{m-1} \to \mathbb{N}_m \)
by
\[ g(i) = \begin{cases} i & \text{when } i < i_0 \\ i + 1 & \text{when } i \geq i_0 \end{cases} \]
and notice that \( g \) is injective (why?).

Now, let \( f_1 = f \circ g : \mathbb{N}_{m-1} \to \mathbb{N}_k \)
and notice that \( f_1 \) is injective as it is the composition
of two injections (proving this is a IH problem)

So, by the inductive hypothesis we have \( m - 1 \leq k \)
and so \( m \leq k + 1 \).

So by induction, we are done.
Corollary: Let \( X \) & \( Y \) be finite, non-empty sets.

If there is an injection \( f : X \rightarrow Y \)
then \(|X| \leq |Y|\)

*Proof is an exercise.*

Example: If we have a number of pigeons
and we place each pigeon in a separate pigeon hole, then

number of pigeons \( \leq \) number of holes.

Contrapositive: If \(|X| > |Y|\) there is no injection \( f : X \rightarrow Y \).

Thus, if \( f : X \rightarrow Y \) is a function, \( \exists x_1, x_2 \), \( x_1 \neq x_2 \), \( f(x_1) = f(x_2) \).

Example: If we have more pigeons than pigeonholes, at least two pigeons
must share a pigeonhole.
Example: If you have a drawer with 10 red and 10 blue socks (unpaired), and you are pulling socks from the drawer without looking, what is the minimum number of socks you must draw to guarantee a pair of the same color?

Answer: Three, by the pigeonhole principle. Take \( X = Y = \{ \text{red, blue} \} \), then \( |Y| = 2 \), and we need \( |X| \geq 3 \) so that we are sure \( \exists x_1, x_2 \) \( f(x_1) = f(x_2) \) while \( x_1 \neq x_2 \).

Some consequences of the pigeonhole principle:

**Definition:** Given a set \( X \), if \( |X| = n \) for a non-negative integer \( n \), then we say \( X \) is finite.

Otherwise, we say \( X \) is infinite.

**Proposition:** Suppose \( f: X \rightarrow \mathbb{N}_n \) is an injection. Then \( X \) is finite and \( |X| \leq n \).

Proof: Exercise.
Proposition: Suppose $X \subseteq Y$, where $Y$ is a finite set. Then $X$ is also finite and $|X| \leq |Y|$. 

Proof: This is a simple corollary of the previous proposition. Indeed, let $n = |Y|$ so there is a bijection $f : \mathbb{N}_n \to Y$. Let $i : X \to Y$ with $i(x) = x$, and note that $i$ is an injection. Thus $f \circ i : X \to Y \to \mathbb{N}_n$ is an injection, so by the previous proposition $X$ is finite and $|X| \leq n$.

Theorem: Let $X, Y$ be non-empty finite sets and let $f : X \to Y$ be a function. Then $f$ is not surjective.

Theorem: Let $X$ & $Y$ be non-empty finite sets with $|X| = |Y|$. Then $f$ is an injection if and only if it is surjective.

The proofs are exercises left to the students.
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Counting Principles

**Theorem (addition principle)**
Suppose \( X \) and \( Y \) are finite sets with \( X \cap Y = \emptyset \). Then \( X \cup Y \) is finite and \( |X \cup Y| = |X| + |Y| \).

**Proof:** We will show that \( |X| + |Y| = |X \cup Y| \), which implies \( X \cup Y \) is finite.

Let \( |X| = n \), \( |Y| = m \). If \( X = \emptyset \), then \( n = 0 \) and \( X \cup Y = Y \), and so \( |X \cup Y| = |Y| = m + n \).

Similarly, the conclusion holds if \( m = 0 \).

Let's consider now the case \( n \neq 0 \) and \( m \neq 0 \).

Note that there are bijections \( f : \mathbb{N}_n \to X \) and \( g : \mathbb{N}_m \to Y \)

and define \( h : \mathbb{N}_{n+m} \to X \cup Y \) via

\[
    h(i) = \begin{cases} 
        f(i) & \text{if } 1 \leq i \leq n \\
        g(i-n) & \text{if } n+1 \leq i \leq n+m
    \end{cases}
\]

and note that \( h \) is bijective (why?).

So \( |X \cup Y| = m + n \).
Corollary: If $X_1, \ldots, X_n$ are pairwise disjoint sets, then
\[ |X_1 \cup X_2 \cup \cdots \cup X_n| = \sum_{i=1}^n |X_i|. \]

Proof: By induction (exercise).

Theorem: (The multiplication principle), Let $X, Y$ be sets.
Suppose $|X| = n$, $|Y| = m$,
then $|X \times Y| = mn$.

Proof: exercise.

The inclusion-exclusion principle

(I) The case of two sets $X$ and $Y$.

Theorem: Let $X$ and $Y$ be finite sets, then
\[ |X \cup Y| = |X| + |Y| - |X \cap Y|. \]
Proof: Recall that $X \cup Y = (X \setminus Y) \cup (X \cap Y) \cup (Y \setminus X)$ and that $X \setminus Y, Y \setminus X, X \cap Y$ are pairwise disjoint.

So by the addition principle

$$|X \cup Y| = |X \setminus Y| + |Y \setminus X| + |X \cap Y|. \quad - (1)$$

However $X = (X \setminus Y) \cup (X \cap Y)$ so

$$|X| = |X \setminus Y| + |X \cap Y| \quad - (2)$$

and similarly $|Y| = |Y \setminus X| + |X \cap Y| \quad - (3)$

Combining (1), (2), (3), we obtain

$$|X \cup Y| = |X| + |Y| - |X \cap Y|. \quad \square$$

(II) General inclusion-exclusion principle.

Theorem: Let $A_1, \ldots, A_n$ be finite sets, and for $I = \{i_1, i_2, \ldots, i_r\} \subseteq \mathbb{N}_n$, define

$$A_I = \bigwedge_{i \in I} A_i = A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r},$$

then

$$\left| \bigcup_{I \subseteq \mathbb{N}_n, I \neq \emptyset} A_I \right| = \sum_{I \subseteq \mathbb{N}_n} (-1)^{|I| - 1} |A_I|.$$
So, for example,

\[ |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3| \]

**Example:** Given 144 tiles that are either triangular or square, red or blue, wooden or plastic. Suppose there are

- 68 wooden tiles
- 69 red tiles
- 75 triangular tiles

and

- 36 red wooden tiles
- 40 triangular wooden tiles
- 38 red triangular tiles

and 23 red wooden triangular tiles.

How many blue plastic square tiles?
Solution: Let \( A_1 = \{ \text{blue tiles} \} \),
\( A_2 = \{ \text{plastic tiles} \} \),
\( A_3 = \{ \text{square tiles} \} \).

We want \( |A_1 \cap A_2 \cap A_3| \), and we’ll get it by the inclusion/exclusion principle, but first we observe that \( (A_1 \cap A_2 \cap A_3)^c = A_1^c \cup A_2^c \cup A_3^c \).

So now, we want
\[
|A_1^c \cup A_2^c \cup A_3^c|.
\]

\[
|A_1^c \cup A_2^c \cup A_3^c| = |A_1^c| + |A_2^c| + |A_3^c| - |A_1^c \cap A_2^c| - |A_1^c \cap A_3^c| - |A_2^c \cap A_3^c|
\]
\[
= 69 + 68 + 75 - 36 - 38 - 40 + 23
\]
\[
= 121.
\]

So we have 121 blue plastic square tiles.