Problem 1.

(a) This statement is false. In order to show that this statement is false, it suffices to show that its negation is true. The negation of this statement reads

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, xy \neq 1.$$ 

Now let $$y \in \mathbb{R}$$. Choose $$x = 0 \in \mathbb{R}$$, then $$xy = 0 \neq 1$$, as required.

(b) This statement is true. Assume that there exists a $$q \in \mathbb{Z}$$ with $$n = 2q + 1$$, then $$n^2 = (2q + 1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1$$, so taking $$p = 2q^2 + 2q$$ satisfies the desired equality $$n^2 = 2p + 1$$.

Problem 2.

(a) Since the codomain of $$g$$ is the domain of $$f$$ we can talk about $$f \circ g$$ and $$f \circ g : \mathbb{R} \to \mathbb{R}$$ is the function given by $$(f \circ g)(x) = f(g(x)) = f(x^3 + 1) = (x^3 + 1)^2$$. Since the codomain of $$f$$ is different from the domain of $$f$$, $$f \circ f$$ is undefined.

(b) We claim that $$f$$ is injective, but not surjective (and consequently also not bijective). To see that $$f$$ is injective, suppose that $$x_1, x_2 \in \mathbb{R}^2$$ satisfy $$f(x_1) = f(x_2)$$, or equivalently $$x_1^2 = x_2^2$$. Since $$x_1, x_2 \geq 0$$ this implies that $$x_1 = x_2$$, i.e. $$f$$ is injective. (This is something you probably know from calculus, but we also saw this on the homework, for example in exercise 3.8.(i))

To see that $$f$$ is not surjective, note that $$−1$$ belongs to the codomain $$\mathbb{R}$$ of $$f$$. However, for any $$x \in \mathbb{R}^2$$, the quantity $$f(x) = x^2$$ is the square of a real number and hence nonnegative, so cannot be equal to $$−1$$. Therefore, $$−1$$ is not in the range of $$f$$.

Problem 3. Suppose that $$x_1, x_2 \in X$$ satisfy $$(g \circ f)(x_1) = (g \circ f)(x_2)$$, or equivalently $$g(f(x_1)) = g(f(x_2))$$. Since $$g$$ is injective we can conclude that $$f(x_1) = f(x_2)$$. Now, since $$f$$ is injective we conclude that $$x_1 = x_2$$, showing that $$g \circ f$$ is injective.

Problem 4. Divide the $$1 \times 1$$ square into 4 squares of $$1/2 \times 1/2$$ and number them 1 through 4:

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Let $$S$$ be the set whose elements are the five selected points and define the function $$f : S \to \{1, 2, 3, 4\}$$ by sending a point $$P \in S$$ to the number of the square it lies in (if it’s on the boundary of two or more such squares, pick any of them). Now, $$S$$ has 5 elements, whereas $$\{1, 2, 3, 4\}$$ has only 4, so by the pigeonhole principle $$f$$ is not injective. Therefore, there exist $$P_1, P_2 \in S$$ with $$P_1 \neq P_2$$ and $$f(P_1) = f(P_2)$$, that is $$P_1$$ and $$P_2$$ both lie in the same $$1/2 \times 1/2$$ square. Suppose that the square is placed in the plane with sides parallel to the $$x$$- and $$y$$-axes. Then each $$P_i$$ has coordinates $$(x_i, y_i)$$ where now $$|x_1 - x_2| \leq 1/2$$ and $$|y_1 - y_2| \leq 1/2$$. Now, the distance between $$P_1$$ and $$P_2$$ equals

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} \leq \sqrt{(1/2)^2 + (1/2)^2} = \sqrt{1/2} = \sqrt{2}/2,$$

as desired.

Problem 5. (As pointed out during the exam, you were asked to find the number of elements in \{1, 2, \ldots, 100\} that are divisible by 3, 5 or both.) Write $$S = \{1, 2, \ldots, 100\}$$. Define sets $$A$$ and $$B$$ by $$A = \{n \in S : 3 \text{ divides } n\}$$ and $$B = \{n \in S : 5 \text{ divides } n\}$$. The question is to find $$|A \cup B|$$. By the inclusion-exclusion principle, $$|A \cup B| = |A| + |B| - |A \cap B|$$. Firstly, $$A = \{3, 6, 9, \ldots, 96, 99\}$$, so $$|A| = 33$$ and $$B = \{5, 10, 15, \ldots, 95, 100\}$$, so $$|B| = 20$$. Now, note that $$A \cap B = \{n \in S : n \in A \text{ and } n \in B\} = \{n \in S : 3 \text{ divides } n \text{ and } 5 \text{ divides } n\} = \{n \in S : 15 \text{ divides } n\}$$.

Therefore, $$A \cap B = \{15, 30, 45, 60, 75, 90\}$$, so $$|A \cap B| = 6$$, and the desired answer is $$33 + 20 - 6 = 47$$. 