Derivatives of higher order:

- Let $E \subseteq \mathbb{R}^n$ be open & $f : E \to \mathbb{R}$ with partial $D_j f$ which are themselves differentiable. Then, the second order partials are defined by

$$D_{ij} f = D_i D_j f \quad \text{where } i,j = 1, \ldots, n$$

We say that $f \in C''(E)$ if each of the $D_{ij} f$ is continuous.

- $f : E \to \mathbb{R}^m$ is $C''$ if each of its $m$ component functions is continuous.

Remark: $D_{ij} f \neq D_{ji} f$ is possible unless both are continuous.

\[ MVT \]
Theorem (MVT). Let $f: E \subset \mathbb{R}^2 \to \mathbb{R}$ and suppose

- $E$ is open
- $D_1 f$ and $D_2 f$ exist at every point in $E$.

Suppose further that $QCE$ is a closed rectangle with sides parallel to the coordinate axes. Let $(a, b)$ & $(a+h, b+k)$ be its opposite vertices and set

$$
\Delta(f, Q) = f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a, b)
$$

Then $\exists (x, y) \in \text{interior}(Q)$ s.t.

$$
\Delta(f, Q) = h k \left( D_2 f \right)(x,y).
$$

Proof. By MVT (in one variable), $\exists \xi \in (a, a+h)$ s.t.

$$
\Delta(f, Q) = f(a+h) - f(a) = f'(\xi) h
$$

where $u(t) = f(t, b+k) - f(t, b)$.

So $\Delta(f, Q) = h (D_1 f(u, b+k) - D_1 f(u, b))$

$\Rightarrow$ by MVT $\Delta(f, Q) = h k \left( D_2 f \left( \xi, b+k \right) \right)$ for some $\xi \in (b, b+k)$.
**Theorem:** Let $E \subset \mathbb{R}^2$ be open & $f : E \to \mathbb{R}$ be such that $D_1 f, D_2 f$ & $D_{21} f$ all exist at every pt in $E$ & $D_{21} f$ is cont’s at $(a,b) \in E$.

Then: $D_{21} f$ exists at $(a,b)$ &

$$
(D_{21} f)(a,b) = (D_{21} f)(a,b).
$$

In particular: $D_{21} f = D_{12} f$ if $f \in C^2(E)$.

**Proof:** Want to show $(D_{21} f)(a,b)$ exists & $= (D_{21} f)(a,b)$

i.e., we want to show that

$$
\lim_{h \to 0} \frac{D_{21} f(a+h,b) - D_{21} f(a,b) - D_{21} f(a,b)}{h} = 0
$$

**(Dayn & Darboux exist)**

Let $\varepsilon > 0$, then $k$ small enough so that

$$
|D_{21} f(x,y) - (D_{21} f)(a,b)| < \varepsilon \quad \forall x, y \in Q := [a, a+h] \times [b, b+k]
$$

$$
\Rightarrow |\frac{\Delta f(Q)}{h} - (D_{21} f)(a,b)| < \varepsilon
$$

(By the previous (MV) theorem)
But \( \Delta(f, Q) = f(a+h, b+k) - f(a+h, b) - f(a+h, b+k) - f(a, b) \)

\[ \Rightarrow \lim_{k \to 0} \frac{\Delta(f, Q)}{k} = D_2 f(a+h, b) - D_2 f(a, b) \]

Fixing \( h \) in \( \mathcal{O} \) & letting \( k \to 0 \) yields

\[ \left| \frac{D_2 f(a+h, b) - D_2 f(a, b) - D_{a+h} f(a, b)}{h} \right| < \epsilon \]

\[ \Rightarrow D_2 f(a, b) = D_{a+b} f(a, b) \]

Differentiation of Integrals

Goals: Derive conditions so that

\[ \frac{d}{dt} \int_a^b y(x,t) \, dx = \int_a^b \frac{\partial y}{\partial t}(x,t) \, dx \]

Remark: not always true!
Theorem: Suppose \( \Psi: [a,b] \times [c,d] \to \mathbb{R} \) 

& \& d is an increasing \( \mathbb{R}^+ \) on \([a,b]\) 
where \( \Psi_t(x) = \Psi(x,t) \in \mathbb{R}(d) \) \( \forall t \in [c,d] \).

Suppose that \( \forall \varsigma \in (c,d), \forall \varepsilon > 0 \) s.t.

\[ |(D^2 \Psi)(x,t) - D^2 \Psi(x,s)| < \varepsilon \]

holds \( \forall x \in [a,b] \) \& \( t \in (s-\delta, s+\delta) \).

Define \( f(t) = \int_a^b \Psi(x,t) \, dx(t) \) \( \forall t \in [c,d] \).

Then \( (D^2 \Psi)(\cdot,s) \in \mathbb{R}(d) \), and \( f'(s) \) exists and satisfies

\[ f'(s) = \int_a^b (D^2 \Psi)(x,s) \, dx(x). \]

Proof:

Let \( \Psi(x,t) = \frac{\Psi(x,t) - \Psi(x,s)}{t-s} \) where \( |t-s| < \delta \).

By the standard MVT, \( \exists \mu \in (s,t) \) s.t.

\[ \Psi(x,t) = (D^2 \Psi)(x,\mu) \]
Then, by \( \forall s \)

\[
|\frac{\partial^2 f(x,s)}{\partial x \partial t} - \frac{\partial^2 f(x,s)}{\partial x \partial s}| < \epsilon \quad \forall x \in [a,b]
\]

\[
\Rightarrow \psi(t) \xrightarrow{t \to s} D_2 f(\cdot,s) \quad \text{uniformly on } [a,b]
\]

But

\[
\frac{F(t) - F(s)}{t - s} = \int_a^t \psi(x,t) - \psi(x,s) \, dx
\]

\[
\psi(x,t) = \psi_t(x) \in \mathcal{R}(a)
\]

\[
\Rightarrow \text{taking limits} \quad \psi'(s) = \int_a^b \frac{D_2 f(x,s)}{1} \, dx
\]

\[\alpha \text{ is monotonically increasing } \& \ g_n \in \text{Range}(\alpha)
\]

\[
\lim_{n \to \infty} \int_a^b g_n \, dx = \lim_{n \to \infty} \int_a^b g_n \, dx
\]

Let's finish our discussion of Chapter 5 with an example of the Implicit Function Theorem.

Let \( f : \mathbb{R}^5 \to \mathbb{R}^2 \)

be given by

\[
\begin{align*}
\begin{cases}
f_1(x_1,x_2,x_3,x_4,x_5) = 2e^{x_1} + x_2x_3 - 4x_4 + 3 \\
f_2(x_1,x_2,x_3,x_4,x_5) = x_2(x_1) - 6x_1 + 2x_3 - x_5
\end{cases}
\end{align*}
\]
and note that \( f(0,1,3,2,7) = 0 \).

Moreover, \( f'(0,1,3,2,7) = [A] = \begin{bmatrix} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1 \end{bmatrix} \)

Note that \( A_x \) is invertible \( \Rightarrow \) by the implicit function theorem \( \exists g: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \), continuously diff.

in the neighborhood of \( (3,2,7) \)

with \( g(3,2,7) = (0,1) \) \& \( f(g(y), y) = 0 \)

Moreover, \( g'(3,2,7) = -A_x^{-1} A_y \)

\[
= \frac{-1}{20} \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 0 \\ 0 & -1 \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{2}{5} & \frac{3}{10} & \frac{3}{20} \\ -\frac{1}{5} & \frac{3}{10} & \frac{1}{10} \end{bmatrix}
\]
Example (Wikipedia :) 

Consider \( f(t) = \int_{0}^{2\pi} \frac{e^{t \cos x} \cos (t \sin x)}{\varphi(x,t)} \, dx \)

\[ \Rightarrow D_2 \varphi = \frac{\partial}{\partial t} \varphi = \ldots \]

\[ f'(t) = \int_{0}^{2\pi} \frac{\partial}{\partial x} \left( e^{t \cos x} \cos (t \sin x) \right) \, dx \]

\[ = \int_{0}^{2\pi} e^{t \cos x} \left( \cos x \cos (t \sin x) - \sin (t \sin x) \sin x \right) \, dx \]

\[ = \frac{1}{t} \int_{0}^{2\pi} e^{t \cos x} \sin (t \sin x) \, dx \]

\[ = \frac{1}{t} \int_{0}^{2\pi} \frac{d}{dx} \left( e^{t \cos x} \sin (t \sin x) \right) \, dx \]

\[ = \frac{1}{t} \left. \left( e^{t \cos x} \sin (t \sin x) \right) \right|_{x=2\pi}^{x=0} \]

\( b) \) Evaluate \( f(0) \) using the definition of \( f \)

\[ \Rightarrow f(0) = 2\pi \]

\( c) \) Conclude that \( \int_{0}^{2\pi} e^{t \cos x} \cos (t \sin x) \, dx = 2\pi \)

Since \( \frac{df}{dt} = 0 \), then \( 0 = \int_{0}^{1} \frac{df}{dt} \, dt = f(1) - f(0) \)
\( \Rightarrow f(1) = f(0) = 2\pi \)

\( \varepsilon \) but \( f(1) \) is \( \int_{-\pi}^{\pi} e^{\cos x} \cos(\sin x) \, dx = 2\pi \).