Math 140C: Midterm 2
Foundations of Real Analysis

• You have 1 hour and 20 minutes. No books and notes are allowed.
• You may quote any result stated in the textbook or in class.
• State carefully the hypothesis and conclusion of any result that you use.
• You may not use homework problems (without proof) in your solutions.
1. (10 points) Let $X$ be a measurable space, in which $\mathcal{M}$ is the $\sigma$-ring of measurable sets and $\mu$ is the measure. Let $\{A_n\}_{n=1}^\infty$ be a sequence of sets such that $A_n \in \mathcal{M}$ and $A_{n+1} \subset A_n$, for all $n \geq 1$. Assume that $\mu(A_1) < +\infty$ and let $A = \cap_{n=1}^\infty A_n$.

(a) (5 points) Prove that $\lim_{n \to \infty} \mu(A_n) = \mu(A)$.

(b) (5 points) Let $f$ be a measurable nonnegative function on $X$ such that $f \in \mathcal{L}(\mu)$ on $X$. Prove that $\lim_{n \to \infty} \int_{A_n} f \, d\mu = \int_A f \, d\mu$. 
2. (10 points) Let $\mathcal{E}$ be the family of elementary subsets of $\mathbb{R}$ and $m : \mathcal{E} \to [0, \infty)$ be the Lebesgue set function. For a set $A \subset \mathbb{R}$, we denote by $m^*(A)$ the outer measure of $A$ corresponding to $m$. For $t \in \mathbb{R}$, we denote by $A + t$ the set $\{x + t | x \in A\}$.

(a) (5 points) Assume that $A \in \mathcal{E}$.
Prove that $m(A + t) = m(A)$, for all $t \in \mathbb{R}$.

(b) (5 points) Assume that $A \subset \mathbb{R}$ is an arbitrary subset.
Prove that $m^*(A + t) = m^*(A)$, for all $t \in \mathbb{R}$.
3. (10 points) Let $X$ be a measurable space, in which $\mathcal{M}$ is the $\sigma$-ring of measurable sets and $\mu$ is the measure. Consider a function $f : X \rightarrow \mathbb{R}$.

(a) (5 points) Assume that the set $\{x \in X \mid f(x) > q\}$ is measurable, for every $q \in \mathbb{Q}$.
Prove that $f$ is measurable.

(b) (5 points) Assume that $f$ is measurable and let $\{f_n : X \rightarrow \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of measurable functions.
Prove that the set of points $x \in X$ such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ is measurable.
4. (10 points) Let $X$ be a measurable space, in which $\mathcal{M}$ is the $\sigma$-ring of measurable sets and $\mu$ is the measure. Let $f : X \to [0, \infty)$ be a measurable function. For $n \geq 1$, let $A_n = \{x \in X \mid n - 1 \leq f(x) < n\}$ and $B_n = \{x \in X \mid f(x) \geq n\}$.

(a) (3 points) Prove that $f(x) \leq \sum_{n=1}^{\infty} n \mathbf{1}_{A_n}(x)$, for every $x \in X$.

(b) (3 points) Assume that $\mu(A_n) \leq \frac{1}{n^3}$, for all $n \geq 1$.

Prove that $f \in L(\mu)$.

(c) (4 points) Assume that $f \in L(\mu)$.

Prove that the sequence $\{n \mu(B_n)\}_{n=1}^{\infty}$ is bounded.
Do not write on this page.

<table>
<thead>
<tr>
<th></th>
<th>out of 10 points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>out of 40 points</td>
</tr>
</tbody>
</table>
