1 Lebesgue Theory

1.1 Set functions

Definition 1.1 A family of sets $\mathcal{R}$ is called a ring if

$$A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R} \text{ and } A \setminus B \in \mathcal{R}.$$  

Thus, a ring is closed under finite unions and set differences (as well as finite intersections).

Definition 1.2 A $\sigma$-ring is a ring that is also closed under countable unions, i.e.,

$$A_n \in \mathcal{R}, n = 1, 2, \ldots \implies \cup_n A_n \in \mathcal{R}.$$  

It can be deduced that a $\sigma$-ring is also closed under countable intersections.

Definition 1.3 A set function on a ring (or $\sigma$-ring) assigns to every element of $\mathcal{R}$ a number (in the extended reals).

Definition 1.4 A set function $f$ is additive if for disjoint sets $A, B \in \mathcal{R}$,

$$f(A \cup B) = f(A) + f(B).$$

Definition 1.5 A set function $f$ is countably additive if for disjoint sets $A_i \in \mathcal{R}, i = 1, 2, \ldots$

$$f(\cup A_i) = \sum_{i=1}^{\infty} f(A_i).$$

Theorem 1.6 Suppose $f$ is a countably additive function on a $\sigma$-ring $\mathcal{R}$. Let $A_n \in \mathcal{R}, n = 1, 2, \ldots$ with $A_1 \subset A_2 \subset \ldots$ and suppose that $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$.  

Then
\[
\lim_{n \to \infty} f(A_n) = f(A).
\]

### 1.2 Construction of the Lebesgue measure

**Definition 1.7** An interval in \( \mathbb{R}^p \) is a set of the form \( I = [a_1, b_1] \times ... \times [a_p, b_p] \).

**Definition 1.8** An elementary set is a finite union of intervals.

**Definition 1.9** For an interval \( I \), define the set function \( m \), via
\[
m(I) := \prod_{i=1}^{p} (b_i - a_i).
\]

**Definition 1.10** For a finite disjoint union of intervals \( I_i \), set
\[
m \left( \bigcup_{i=1}^{n} I_i \right) := \sum_{i=1}^{n} m(I_i).
\]

**Definition 1.11** Denote by \( \mathcal{E} \) the collection of all elementary subsets of \( \mathbb{R}^p \).

**Remark 1.12** \( \mathcal{E} \) is a ring, but not a \( \sigma \)-ring. Elements of \( \mathcal{E} \) can be decomposed as finite, disjoint unions of intervals.

**Remark 1.13** The function \( m \) defined above is additive on \( \mathcal{E} \).

**Definition 1.14 (Regularity)** A non-negative, additive set function \( f \) defined on \( \mathcal{E} \) is regular if
\[
\forall A \in \mathcal{E}, \epsilon > 0, \exists F, G, \in \mathcal{E}, \text{ where } F \text{ is closed and } G \text{ is open},
\]
and
\[
F \subset A \subset G,
\]
with
\[
f(G) - \epsilon \leq f(A) \leq f(F) + \epsilon.
\]

**Example 1.15** The set function \( m \) defined above is regular on \( \mathcal{E} \).

**Definition 1.16 (Outer measure)** Let \( \mu \) be a non-negative, additive, finite, regular set function defined on \( \mathcal{E} \). The outer measure of \( E \subset \mathbb{R}^p \) is given by
\[
\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n)
\]
where the infimum is taken over all countable covers of $E$ by elementary open sets.

**Fact 1.17** The following are simple to deduce from the definition:

$$
\mu^*(E) \geq 0,
$$

$$
E_1 \subset E_2 \implies \mu^*(E_1) \leq \mu^*(E_2).
$$

**Theorem 1.18** Let $\mu$ be finite, non-negative, additive, regular, then $\mu^*$ agrees with $\mu$ on elementary sets, and it is countably sub-additive. That is,

(a) $A \in \mathcal{E} \implies \mu^*(A) = \mu(A),$

(b) $E = \bigcup_{i=1}^{\infty} E_i \implies \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$

**Definition 1.19 (Convergence)** We say that a sequence of sets $A_n$ converges to $A$ if

$$
\lim_{n \to \infty} d(A, A_n) = 0,
$$

where for two sets $A$ and $B$ we define

$$
d(A, B) := \mu^* ((A \setminus B) \cup (B \setminus A)).
$$

Notice that the notion of coverage above depends on the choice of $\mu$.

**Definition 1.20 (Finitely $\mu$-measurable sets)** If there is a sequence of elementary sets converging to $A$, we say $A$ is finitely $\mu$-measurable; we write $A \in M_F(\mu)$.

**Definition 1.21 ($\mu$-measurable sets)** If $A$ is a countable union of finitely $\mu$-measurable sets, we say that $A$ is $\mu$-measurable; we write $A \in M(\mu)$.

**Theorem 1.22** $M(\mu)$ is a $\sigma$-ring and $\mu^*$ is countably additive on $M(\mu)$.

Thus we may now replace $\mu^*(A)$ by $\mu(A)$ – and we can call $\mu$ a measure. When $\mu = m$, we call it the Lebesgue measure.

**Definition 1.23** A Borel set is a set that can be obtained via countable unions, countable intersections, and/or set differences, complements, of open sets.

**Remark 1.24** The collection of Borel sets in $\mathbb{R}^p$ is a $\sigma$-ring. In fact, it is the smallest $\sigma$-ring containing all open sets.

**Remark 1.25** Every $A \in M(\mu)$ is the union of a Borel set and a set of measure zero.
2 Measure spaces

Definition 2.1 Let $X$ be some set. If there exists a $\sigma$-ring $\mathcal{M}$ of subsets of $X$ (called measurable sets) and a non-negative countable additive set function $\mu$ (called a measure) defined on $\mu$, then $X$ is called a measure space. We often write $(X, \mathcal{M}, \mu)$ to identify the $\sigma$-ring and measure associated with $X$.

Definition 2.2 We say that a function $f$ defined on a measurable space $\mathcal{M}$ is measurable if 
$$\{x | f(x) > a\}$$
is measurable (i.e., belongs to $\mathcal{M}$) for all $a$.

Theorem 2.3 The following are equivalent:
- $\{x | f(x) > a\}$ is measurable for all real $a$.
- $\{x | f(x) \geq a\}$ is measurable for all real $a$.
- $\{x | f(x) < a\}$ is measurable for all real $a$.
- $\{x | f(x) \leq a\}$ is measurable for all real $a$.

Theorem 2.4 The inf, sup, lim inf, and lim sup of a sequence of measurable functions are measurable. The limit of a converging sequence of measurable functions is measurable.

Theorem 2.5 Suppose that $f$ and $g$ are measurable functions, then
- $|f|$ is also measurable
- $\max(f, g), \min(f, g), f^+ = \max(f, 0), f^- = -\min(f, 0)$ are measurable.

Theorem 2.6 If $f, g$ are measurable real-valued functions on $X$, and $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then $h$ defined via 
$$h(x) = F(f(x), g(x)), x \in X$$
is measurable. For example, this implies the sum and product of measurable functions is measurable.

Remark 2.7 Notice that the way we define measurable functions does not really require a measure, only a $\sigma$-ring $\mathcal{M}$. 

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