Newton's method for root-finding

**Skeptic:** Quadratic convergence when close to a root. (Faster than bisection)

**Issue:** Not guaranteed always to converge

- **Want to find a zero of some function $f$**

- **Let $r$ be a zero of $g$ ($\iff g(r) = 0$)**

- **If the Taylor series makes sense, i.e., $f''$ exists, and if our current guess for the zero is $x = r - h$**

  $f(r) = f(x + h) = f(x) + h f'(x) + \frac{h^2 f''(x)}{2}

  f(r) = 0 \quad \text{Taylor linear approximation} \quad O(h^2)$

So: $0 = f(x) + h f'(x) + O(h^2)$

For small $h$, we approximate: $0 \approx f(x) + h f'(x)$

So $h \approx \frac{-f(x)}{f'(x)}$ (Remember, we know $x$)

So if we know $h$, we know

$r = h + x$ 1

and $r \approx x - \frac{f(x)}{f'(x)}$
Now, our new approximation is \( x \leftarrow x - \frac{g(x)}{g'(x)} \).

That is we have the iteration

\[
x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}
\]

**Simple Pseudo-code**

INPUT \( x, M \)

\( y \leftarrow g(x) \)

FOR \( k = 1 \) to \( M \) DO

\( x \leftarrow x - y/g'(x) \)

\( y \leftarrow g(x) \)

END DO
INPUT $x_0, M, S, \varepsilon$

$v \leftarrow g(x_0)$

IF $|v| < \varepsilon$ THEN STOP

FOR $k = 1$ to $M$ DO

$x_1 \leftarrow x_0 - v / f'(x_0)$

$v \leftarrow g(x_1)$

IF $|x_1 - x_0| < S$ or $|v| < \varepsilon$ THEN STOP

$x_0 \leftarrow x_1$

END DO
Interpretation:

\[ f(x) = f(x_n) + f'(x_n)(x-x_n) + \ldots \]

\[ \ell(x) = f(x) + f'(x_n)(x-x_n) \]

\( x_{n+1} \) is a root of \( \ell(x) \).

Issue: If \( x_0 \) is not very close to a zero, or if the graph of \( f \) is "not nice", Newton's method may fail.
Error Analysis:

Define the error at $n$'th step: $e_n = x_n - r$

Assume $f \in C^2(\mathbb{R})$ & $f(r) = 0$ but $f'(r) \neq 0$ (simple zero)

Then:  

$$e_{n+1} = x_{n+1} - r$$

$$= x_n - \frac{f(x_n)}{f'(x_n)} - r \quad \text{(by defn of Newton's method)}$$

$$= x_n - r - \frac{f(x_n)}{f'(x_n)}$$

$$\frac{e_n}{e_n} = e_n - \frac{f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)} \quad (1)$$

But $f(r) = 0 = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{e_n^2 f''(y_n)}{2}$

For some $y_n$ between $x_n$ & $r \Rightarrow e_n f'(x_n) - f(x_n) = \frac{e_n^2 f''(y_n)}{2} \quad (2)$
So we need to understand when $\frac{f''(x_0)}{f'(x_0)}$ is small.

Let $c(\delta) = \frac{1}{2} \max_{|x - r| < \delta} \left| \frac{f''(x)}{|f'(x)|} \right|$ for $\delta > 0$.

Pick $\delta$ small enough so $\min_{|x - r| > 0} |f'(x)| > 0$ and $\delta c(\delta) < 1$.

Let $p = \delta c(\delta)$ and suppose $|x_0 - r| < \delta$.

\[ |e_0| < \delta \Rightarrow |x_0 - r| < \delta \Rightarrow \frac{1}{2} \left| \frac{f''(x_0)}{f'(x_0)} \right| < c(\delta) \]

\[ |e_1| < \delta c(\delta) = |e_0| |e_0| c(\delta) \leq p |e_0| < |e_0| < \delta \]

Since $|e_1| = |x_0 - r| < \delta$ we can do the same thing for $e_2$.

\[ |e_2| < p |e_2| < p^2 |e_0| \]

And

\[ |e_n| < p^n |e_0| \xrightarrow{n \to \infty} 0 \quad \text{(because } p < 1 \text{)} \]
**Theorem**: Suppose $f \in C^2(\mathbb{R})$, $f(r) = 0$, $f'(r) \neq 0$. Then $\exists \delta > 0$ and a $\rho < 1$, such that if $|x_0 - r| < \delta$ Newton's method started at $x_0$ yields

$$|x_{n+1} - r| \leq \rho (x_n - r)^2 \quad \text{for} \ n \geq 0.$$ 

**Theorem**: If $f \in C^2(\mathbb{R})$, increasing, convex ($f'' > 0$), has a zero then the zero is unique & the Newton iteration will converge to it. From any starting point.

**Ex**: $f(x) = x^2 - R$ has a root at $x = \sqrt{R}$ so we can use Newton's method to find it

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - R}{2x_n} = \frac{1}{2}(x_n + \frac{R}{x_n})$$

Recall that Newton's iteration was derived from linearizing the function $f$ (i.e. using 1st order Taylor approx).
Same idea for functions of many variables.

\[
\begin{align*}
\text{want to solve: } \begin{cases}
L_1(x_1, \ldots, x_n) = 0 \\
L_2(x_1, x_2, \ldots, x_n) = 0 \\
\vdots \\
L_n(x_1, x_2, \ldots, x_n) = 0
\end{cases}
\end{align*}
\]

\[
\iff \begin{pmatrix} f_1 & f_2 & \cdots & f_n \end{pmatrix}^T = 0
\]

\[
\Rightarrow F(X) = 0 \quad \text{where } X = (x_1, x_2, \ldots, x_n)^T
\]

Starting at an estimate \(X\):

\[
F(X + H) \approx F(X) + F'(X)H
\]

If \(X + H\) is a root, then \(F(X + H) = 0 \approx F(X) + F'(X)H\)

So \(H = -\left(F'(X)\right)^{-1}F(X)\)

\(\text{can be expensive to invert large matrices}\)

So we prefer to solve the system by, e.g.,

\(\text{Gaussian elimination}\)
Newton's method: \( X^{(n+1)} = X^{(n)} + H^{(n)} \)

where \( H^{(n)} \) satisfies

\[ F'(X^{(n)}) H^{(n)} = -F(X^{(n)}) \]

Equivalently: \( X^{(n+1)} = X^{(n)} - (F'(X^{(n)}))^{-1} F(X^{(n)}) \)

Example: To solve: \( \begin{cases} xy = z^2 + 1 \\ x^2 + y^2 = x^2 + 2 \\ e^x + z = e^3 + 3 \end{cases} \)

we set up \( F(X) = \begin{pmatrix} f_1(x, y, z) \\ f_2(x, y, z) \\ f_3(x, y, z) \end{pmatrix} = \begin{pmatrix} xy - z^2 - 1 \\ x^2 + y^2 - x^2 - 2 \\ e^x + z - e^3 - 3 \end{pmatrix} \)

\[ F'(X) = \begin{pmatrix} y & x & -2z \\ yz - 2x & x^2 + y & zy \\ e^x & -e^y & 1 \end{pmatrix} \]
Sequent Method

**Newton:** \[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

**Sequent:** \[ x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} \] (approx. to \( f'(x_n) \))

\[ n \geq 1 \]

Remark: Need two initial points

Remark: \[ |x_{n+1}| \approx A |x_n|^{(\sqrt{5})/2} \]

\[ A = \left| \frac{8''(r)}{2 \cdot 8'(r)} \right|^{\frac{\sqrt{5} - 1}{2}} \]