Fixed points & Functional iteration

Newton's method: \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \)

(call this \( F(x_n) \))

So, Newton's method is a functional iteration \( x_{n+1} = F(x_n) \)

Suppose (in general now) that \( \lim_{n \to \infty} x_n \) exists & \( \lim_{n \to \infty} x_n = L \)

Suppose also that \( F \) is continuous
Then \[ \lim_{n \to \infty} F(x_n) = F\left( \lim_{n \to \infty} x_n \right) = F(\delta) \]

1. (here \( x_{n+1} = F(x_n) \))

\[ \lim_{n \to \infty} x_{n+1} = \delta \]

So \( \delta = F(\delta) \).

Such a point is called a fixed pt. of \( F \).

Very important & interesting problems (e.g. in optimization, algorithm design, diff. equations ....) can be reduced to finding the fixed pts. of a function \( F \).

Simple (but important case):

\( F : C \to C \) where \( C \) is a closed subset of \( IR \).
**Contractive mapping:**

\[ |F(x) - F(y)| \leq \lambda |x - y| \]

for some \( \lambda < 1 \)

(can you see why it's called contractive?)

**Theorem:** Let \( C \subseteq \mathbb{R} \) be closed. If \( F: C \to C \) is contractive, then \( F \) has a unique fixed point \( s \). Moreover,

\[ s = \lim_{n \to \infty} x_{n+1} \quad \text{where} \quad x_{n+1} = F(x_n) \]

for any starting point \( x_0 \in C \).

**Proof:** Want to show that \( (x_n) \) converges.

But \( x_n = x_0 + (x_1 - x_0) + \cdots + (x_n - x_{n-1}) \)

\[ = \left( \sum_{i=1}^{n} (x_i - x_{i-1}) \right) + x_0 \]

Want this sequence to converge as \( n \to \infty \).
It suffices to show that \( \sum_{i=1}^{\infty} |x_i - x_{i-1}| \) converges.

But

\[
|x_{i+1} - x_i| = |F(x_i) - F(x_{i-1})|
\]

is a contractive map \( \lambda \) such that

\[
|F(x_i) - x_{i-1}| \leq \lambda \cdot |x_i - x_{i-1}|
\]

\[
\Rightarrow x_i \leq \lambda x_0
\]

So

\[
\sum_{i=1}^{\infty} |x_i - x_{i-1}| \leq \sum_{i=1}^{\infty} \lambda i |x_i - x_0| = \frac{1}{1-\lambda} |x_1 - x_0|
\]

\( \Rightarrow \) the sequence converges so that \( S = \lim_{n \to \infty} x_n \)

and note that \( S = F(S) \).

To see that \( S \) is unique, suppose \( t \) is a different fixed pt. so \( t = F(t) \)

\[
|t - S| = |F(t) - F(S)| \leq \lambda |t - S|
\]

But \( \lambda < 1 \) so \( t = S \).
Exercise: Let \( F(x) = a + b \sin(x) \) for some \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \). For what values of \( a \) and \( b \) is \( F \) contractive? In that case, write an iteration to find the fixed point of \( F \).

Order of Convergence: Suppose \( \frac{x_{n+1}}{F(s)} = s \)

\[
\text{let } e_n = x_n - s
\]

(then \( \lim_{n \to \infty} e_n = 0 \)) if the order of convergence is the smallest integer \( k \) such that \( F^{(k)}(s) \neq 0 \).