A spline function consists of polynomial pieces on subintervals. They are joined together in a way that satisfies continuity conditions.

\[
\begin{align*}
&\text{on } [t_{i-1}, t_i), S \text{ is a polynomial of degree } \leq k \\
&\text{S has a continuous } (k-1)^{\text{st}} \text{ derivative on } [t_0, t_n]
\end{align*}
\]
e.g. Spline of degree 0: piecewise constant
Spline of degree 1: piecewise linear

**Cubic Splines** (Splines with degree $k=3$)

Denote by $S(x)$ the cubic spline interpolating some data $(x_i, y_i)$, $0 \leq i \leq n$

\[
S(x) = \begin{cases} 
S_0(x) & x \in [t_0, t_1] \\
S_1(x) & x \in [t_1, t_2] \\
\vdots & \\
S_{n-1}(x) & x \in [t_{n-1}, t_n]
\end{cases}
\]

so $S(t_i) = S_i(t_i) = y_i$ for $i = 1, \ldots, n-1$

we also ask that $S'$ and $S''$ are continuous

Is this possible?

- we have 2 equations per $S_i$: $S_i(t_i) = y_i$ and $S_i(t_{i+1}) = y_{i+1}$

$\Rightarrow 2n$ equations

- we also have $S_i'(t_i) = S_{i-1}'(t_i)$ for $i = 1, \ldots, n-1$

$\Rightarrow 2(n-1)$ equations $S_i''(t_i) = S_{i-1}''(t_i)$ for $i = 1, \ldots, n-1$
\( \Rightarrow \) Total of \( 4n-2 \) equations
But each \( S_i(x) \) has 4 degrees of freedom
\((a+bx+cx^2+dx^3)\)
\( \Rightarrow 4n\)-unknowns

We're left with 2 degrees of freedom.

Ok, let's now figure out the \( S_i \)'s.

- Let \( z_i = S_i''(t_i) \) 
  \( (z_i = \lim_{x \to t_i} S''(x)) \)
  \( i = 1, \ldots, n-1 \)

Moreover \( S_i''(x) \) is a linear function
with \( S_i''(t_i) = z_i \), \( S_i''(t_{i+1}) = z_{i+1} \)

So

\[ S_i''(x) = \frac{t_{i+1} - x}{t_{i+1} - t_i} \cdot z_i + \frac{x - t_i}{t_{i+1} - t_i} \cdot z_{i+1} \]
To find \( C \) and \( D \):
\[ S_i(t_i) = y_i \quad \text{and} \quad S_i(t_{i+1}) = y_{i+1} \]

\[ \Rightarrow C = \left( \frac{y_{i+1} - z_i h_i}{h_i} \right) \quad \text{and} \quad D = \frac{y_i - z_i h_i}{c} \]

Now, to find the \( z_i \)'s, note that \( S_i'(t_i) = S_i'(t_i) \)

Differentiate the expressions for \( S_i(x) \), \( S_{i-1}(x) \) and plug in \( t_i \):

\[ h_{i-1} z_{i-1} + 2(h_i + h_{i-1}) z_i + h_i z_{i+1} = \frac{6}{h_i} (y_{i+1} - y_i) \]
\[ - \frac{6}{h_{i-1}} (y_i - y_{i-1}) \]

for \( i = 1, \ldots, n-1 \)

\[ \Rightarrow n+1 \text{ unknowns (} z_0 - z_n \text{)} \]
\[ n-1 \text{ equations} \]

Simply set \( z_0 = 0 \), \( z_n = 0 \) and solve for the rest. \( \Rightarrow \) natural cubic spline
**Theorem:** Let \( f \in C^2[a,b] \) & let

\[
a = t_0 < t_1 < t_2 < \cdots t_n = b.
\]

If \( S \) is the natural cubic spline interpolating \( f \) at \( t_i, \ i = 0, \ldots, n-1 \)

then

\[
\int_a^b (S''(x))^2 \, dx \leq \int_a^b f''(x)^2 \, dx
\]

\( \text{Among all } g \text{'s that fit the data } S \text{ is the "smoothest".} \)

**Proof**

Let \( g := f - S \iff f = g + S \)

\[
\Rightarrow \int_a^b (f''(x))^2 \, dx = \int_a^b (g''(x))^2 \, dx + \int_a^b (S''(x))^2 \, dx + 2 \int_a^b g''(x) S''(x) \, dx
\]

\[
\geq 0 \quad \text{?} \quad \geq 0
\]

If we can show that \( \int_a^b g''(x) S''(x) \, dx \) is \( \geq 0 \) we are done.

\[
\int_a^b g''(x) S''(x) \, dx = \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} g''(x) s''(x) \, dx
\]

Integration by parts: \( (g'S') = g''S'' + g'S'' \)
\[
\Rightarrow \sum_{t_{i-1}}^{t_i} g^{''}(s) = \sum_{t_{i-1}}^{t_i} (g^{'}(s))' - \sum_{t_{i-1}}^{t_i} g^{''}(s) = (g^{'}(s))'(t_i) - (g^{'}(s))'(t_{i-1}) - \sum_{t_{i-1}}^{t_i} g^{''}(s)
\]

\[
\Rightarrow \sum_{i=0}^{n} \int_{x}^{t_i} g^{''}(x) s''(x) \, dx = \sum_{i=0}^{n} [g^{'}(s)(t_i) - g^{'}(s)(t_{i-1})] - \sum_{i=0}^{n} \sum_{t_{i-1}}^{t_i} g^{'}(s)
\]

\[
= - \sum_{i=0}^{n} c_i \int_{t_{i-1}}^{t_i} g^{'}(x) \, dx = - \sum_{i=0}^{n} c_i [g(t_i) - g(t_{i-1})]
\]

\[
\Rightarrow (t_i) - S(t_i) \quad \downarrow
\]

\[
\Rightarrow 0
\]

\[
\Rightarrow 0 !
\]