# Math 174, Fall 2018
## Review for Midterm 1
## (Based on problems originally prepared by Marc Loschen)

---

**Problem 1.** Let $f(x) = e^x$.

(a) Find the Taylor Series for $f$ about the point $x = 0$. Include two or three versions of the remainder term.

(b) Find the minimum number of terms needed to compute $e$ with an error less than $10^{-8}$.

(c) Find the minimum number of terms needed to compute $e^2$ with an error less than $2^{-m}$. Your answer should be an inequality.

(d) Suppose we use the Taylor polynomial of degree $n = 2$ to compute $e$. Determine the error bound given by this approximation.

**Problem 2.** Find the order of convergence for each of the following sequences.

(a) $x_n = 3^{-n} + 1$

(b) $x_{n+1} = \frac{1}{2}x_n$, with $x_0 \in \mathbb{R}$

(c) $x_{n+1} = \frac{1}{2}x_n^p$, with $x_0 \in \mathbb{R}$ and $p \geq 1$

(d) $x_{n+1} = \frac{1}{2}(x_n^2 + 1)$, with $x_0 \in (0, 1)$.

(e) $x_n = \dfrac{1}{n!}$

(f) $x_{n+1} = x_n \left(3 - \dfrac{x_n}{a}\right)$, where $a > 0$.

(g) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$, where $a > 0$.

**Problem 3.** Prove the following statements.

(a) $5n^3 + 9n^2 + 1 = \mathcal{O}(n^3)$

(b) $5n^3 + 9n^2 + 1 = \mathcal{O}(n^4)$

(c) $5n^3 + 9n^2 + 1 \neq \mathcal{O}(n^2)$

(d) $\frac{1}{n^2} = o(\frac{1}{n})$



Last Updated: 10/27/2018

(e) $\frac{1}{n^2} \neq o(\frac{1}{n^2})$

**Problem 4.** Let $\{x_n\}$, $\{y_n\}$ and $\{\alpha_n\}$ be sequences in $\mathbb{R}$. Prove the following statements.

(a) $x_n = x + o(1)$ if and only if $\lim_{n\to\infty} x_n = x$.

(b) If $x_n = \mathcal{O}(\alpha_n)$, then $cx_n = \mathcal{O}(\alpha_n)$, where $c$ is a constant.

(c) If $x_n = \mathcal{O}(\alpha_n)$ and $y_n = \mathcal{O}(\alpha_n)$, then $x_n + y_n = \mathcal{O}(\alpha_n)$.

(d) If $x_n = o(\alpha_n)$ and $y_n = o(\alpha_n)$, then $x_n + y_n = o(\alpha_n)$.

(e) If $x_n = \mathcal{O}(\alpha_n)$, $y_n = \mathcal{O}(\alpha_n)$, and $\alpha_n \to 0$, then $x_n y_n = o(\alpha_n)$.

**Problem 5.** Let $F$ be a $C^1$ function, and suppose the sequence $\{x_n\}$ is defined by $x_{n+1} = F(x_n)$. Assume $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. Use the Mean Value Theorem to show that $x_{n+2} - x_{n+1} = O(x_{n+1} - x_n)$.

**Problem 6.** Let $[a, b]$ be an interval in $\mathbb{R}$. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is a function such that $f(a)$ and $f(b)$ have different signs, and let $r$ be a zero of $f$, i.e. $f(r) = 0$. Prove that the error $e_n$ in approximating $r$ with the Bisection Method is bounded as follows:

$$|e_n| \leq 2^{-(n+1)}(b - a)$$

**Problem 7.** Consider running the Bisection method with the starting interval $[1, 4]$. Determine the number of steps $n$ that need to be taken to guarantee the error bound $|r - c_n| \leq 10^{-8}$.

**Problem 8.** Consider this problem: we are trying to approximate the square root of 3.

(a) Formulate this problem as a root-finding problem.

(b) Find two numbers $a, b$ such that $a^2 < 3 < b^2$. Then, considering the interval $[a, b]$, find the error given by the Bisection method after $n$ steps.

(c) Derive the Newton iteration for this problem.

(d) Derive the Secant iteration for this problem.

**Problem 9.** Devise a Newton iteration to compute the $n$th root of a positive number $\alpha$.

**Problem 10.** Perform two iterations of Newton's method on the following problems.

(a) $x^2 - 5x + 6 = 0$, with $x_0 = 1$

(b) $4 - x^4 = x^3$, with $x_0 = 1$

(c) $e^{-x} - x = 0$, with $x_0 = 0$

**Problem 11.** Perform two iterations of the Secant method on the following problems.

(a) $x^2 - 5x + 6 = 0$, with $x_0 = 0, x_1 = 1$

(b) $4 - x^4 = x^3$, with $x_0 = 1, x_1 = 2$

(c) $e^{-x} - x = 0$, with $x_0 = 0, x_1 = 1$

**Problem 12.** Do the following functions have a fixed point in the indicated interval? Provide an argument in either case.

(a) $F(x) = \frac{1}{8}(x-2)^2$ in $[0, 1]$

(b) $F(x) = e^{-x} - x$ in $[0, 1]$

(c) $F(x) = \sin(x) - x$ in $[0, 1]$

**Problem 13.** Consider the fixed-point problem $F(x) = x$. Devise a Newton iteration for this problem (i.e., for finding the fixed point).

**Problem 14.** Given an interval $[a, b]$, let $F \in C^1([a, b])$ and suppose $|F'(x)| \leq 1$ on $[a, b]$ and that $F(x) \in [a, b]$ for all $x \in [a, b]$. Prove that $F$ has a fixed point in $[a, b]$.

**Problem 15.** Consider the following data tables:

(I)

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $y$ | 10 | 1 | 19 |

(II)

| $x$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $y$ | 1 | $-2$ | 8 |

(III)

| $x$ | $-1$ | 1 | 4 | 5 |
|---|---|---|---|---|
| $y$ | 31 | 6 | 0 | 2 |

(IV)

| $x$ | 0 | 8 | 2 | 4 |
|---|---|---|---|---|
| $y$ | 22 | 0 | 0 | $-14$ |

(a) Find the Lagrange form of the interpolating polynomial for each data set.

(b) Find the Newton form of the interpolating polynomial for each data set.

(c) Verify that your answers to (a) and (b) simplify to the same polynomial.

(d) Add the point $(3, 1)$ to each data table above, and find the interpolation polynomial for the modified data sets.

**Problem 16.** Consider the following data table:

| $x$ | 0 | 2 | 3 |
|---|---|---|---|
| $y$ | 13 | 9 | $-20$ |

(a) Find the Lagrange form of the interpolating polynomial for this data.

(b) Find the Newton form of the interpolating polynomial for this data.

(c) Verify that your answers to (a) and (b) simplify to the same polynomial.

**Problem 17.** Consider the following data table:

| $x$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $y$ | 12 | 6 | 4 |

(a) Find the Lagrange form of the interpolating polynomial for this data.

(b) Find the Newton form of the interpolating polynomial for this data.

**Problem 18.** Consider the following data table:

| $x$ | 0 | 1 | 4 |
|---|---|---|---|
| $y$ | 0 | 1 | 64 |

(a) Find the Lagrange form of the interpolating polynomial for this data.

(b) Find the Newton form of the interpolating polynomial for this data.