Math 174, Fall 2018
Review for Midterm 1
(Based on problems originally prepared by Marc Loschen)

Problem 1. Consider the following data tables:

(I)

x	1	4	5
$f(x)$	27	0	-1
$f'(x)$	0	9	-3

(II)

x	-1	0	2	7
$f(x)$	-12	24	0	18
$f'(x)$		10		

(III)

x	0	1	2
$f(x)$	4	-1	12
$f'(x)$	-2	3	
$f''(x)$		8	

(IV)

x	a	b	c
$f(x)$	$f(a)$	$f(b)$	$f(c)$
$f'(x)$	$f'(a)$	$f'(b)$	$f'(c)$
$f''(x)$	$f''(a)$	$f''(b)$	$f''(c)$
$f'''(x)$	$f'''(a)$	$f'''(b)$	$f'''(c)$

(a) Find the Hermite interpolating polynomial for the data sets (I), (II), and (III).

(b) For data set (IV), compute the divided-difference table (you can leave the values as divided-differences, i.e. $f[x_i, x_j]$). Substitute derivative values where appropriate.

Problem 2. Consider the following data table:

x	x_0	x_1
$f(x)$	a_0	a_1
$f'(x)$	b_0	

(a) Find the Hermite interpolating polynomial $p(x)$ for this data.

(b) Compute $p(x_0)$, $p(x_1)$, and $p'(x_0)$. Explain why your answers make sense.

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Problem 3. Let $t_0 < t_1 < t_2 < t_3 < t_4$ be knots, and let $S(x)$ be the spline function that interpolates these knots, where

$$S(x) = \begin{cases} S_0(x), & x \in [t_0, t_1] \\ S_1(x), & x \in [t_1, t_2] \\ S_2(x), & x \in [t_2, t_3] \\ S_3(x), & x \in [t_3, t_4] \end{cases}$$

(a) What conditions does S need to satisfy in order to be a *linear spline*?

(b) What conditions does S need to satisfy in order to be a *quadratic spline*?

(c) What conditions does S need to satisfy in order to be a *cubic spline* (not necessarily natural)?

(d) What conditions does S need to satisfy in order to be a *natural cubic spline*?

Problem 4. Consider the function

$$S(x) = \begin{cases} 1 - 4x + x^3, & x \in [0, 2] \\ a + b(x-2) + c(x-2)^2 + d(x-2)^3, & x \in [2, 5] \end{cases}$$

Find a, b, c, d so that $S(x)$ is a natural cubic spline on $[0, 5]$.

Problem 5. Determine whether the following function is a cubic spline:

$$f(x) = \begin{cases} 2(x+1) + (x+1)^3, & x \in [-1, 0] \\ 3 + 5x + 3x^2, & x \in [0, 1] \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3, & x \in [1, 2] \end{cases}$$

Problem 6 (Math 274 students)**.** Let $f \in C^2([a, b])$ and let $a = t_0 < t_1 < \cdots < t_n = b$ be knots in $[a, b]$. Suppose $S(x)$ is the *natural* cubic spline that interpolates f at these knots $\{t_i\}_{i=0}^n$.

(a) Define $g(x) := f(x) - S(x)$. Show that $\displaystyle\int_a^b S''(x) g''(x)\, dx \geq 0$. (Hint: use integration by parts.)

(b) Using part (a), prove:

$$\int_a^b [S''(x)]^2\, dx \leq \int_a^b [f''(x)]^2\, dx$$

Problem 7. Consider the knots $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$. Find the natural cubic spline function on these knots that interpolates the data points $(-1, 13)$, $(0, 7)$, and $(1, 9)$.

Problem 8. Consider the function

$$S(x) = \begin{cases} x^3, & x \in [0,1] \\ \frac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c, & x \in [1,3] \end{cases}$$

Find a, b, c so that $S(x)$ is a cubic spline on $[0, 3]$. Is $S(x)$ a natural cubic spline?

Problem 9.

(a) Derive an approximation to the first derivative $f'(x)$ using the points x, $x+h$, and $x+2h$. Give the exact error term for this approximation.

(b) Derive an approximation to the first derivative $f'(x)$ using the points $x+h$ and $x-h$. Give the exact error term for this approximation.

(c) Derive an approximation to the second derivative $f''(x)$ using the points x, $x+h$, and $x+2h$. Give the exact error term for this approximation.

(d) Derive an approximation to the second derivative $f''(x)$ using the points $x-h$, x, and $x+h$. Give the exact error term for this approximation.

Problem 10.

(a) Derive the finite difference $f'(x) \approx \dfrac{f(x+2h) - f(x-h)}{3h}$

(b) Using Taylor series, write out the first few terms of the error expansion.

(c) Perform one step of Richardson extrapolation to derive a numerical method of higher order.

Problem 11. Suppose we are trying to approximate a quantity L using a numerical method $\phi(h)$. Suppose further that L is given exactly by $L = \phi(h) + \sum_{j=1}^{\infty} a_j h^j$, where $a_j \in \mathbb{R}$.

(a) Based on the given formula for L, what is the order of the numerical method $\phi(h)$?

(b) Perform two steps of Richardson Extrapolation to derive a numerical method of higher order.

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Problem 12. (We did this in class.)

(a) Give the recursive definition for $T_n(x)$, the Chebyshev polynomials of the first kind.

(b) Show that $\cos((n+1)\theta) = 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta)$. Hint: use the the identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$.

(c) Use part (b) to prove that $T_n(x) = \cos(n\cos^{-1}(x))$ for all $n \geq 0$.

Problem 13. (Math 274 students) Prove that $\displaystyle\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = 0$ unless $n = m$. Hint: use the trig substitution $x = \cos(\theta)$. (Remark: This shows that the Chebyshev polynomials of the first kind are an *orthogonal* family of polynomials.)

Problem 14.

(a) Give the closed-form formula for computing $U_{n+1}(x)$, the Chebyshev polynomials of the second kind.

(b) Prove that these polynomials are generated recursively by $\begin{cases} U_0(x) = 1, \quad U_1(x) = 2x \\ U_{n+1} = 2xU_n - U_{n-1} \end{cases}$ for $n \geq 1$.

Problem 15. Let T_n be the Chebyshev polynomials of the first kind, and let U_n be the Chebyshev polynomials of the second kind. Prove that $T_n' = nU_{n-1}$.

Problem 16. Consider the integral $\int_0^1 f(x)\,dx$. For each set of nodes below, derive a quadrature rule to approximate this integral. You should derive each formula twice: once using Lagrange interpolation, and once using the method of undetermined coefficients.

(a) $\{\frac{1}{3}, \frac{2}{3}\}$, exact for all polynomials of degree ≤ 2

(b) $\{0, \frac{1}{2}, 1\}$, exact for all polynomials of degree ≤ 3

(c) $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, exact for all polynomials of degree ≤ 3

(d) Transform each formula above into one for integration over $[a, b]$.

Problem 17. Determine the values of A, B, C that make the formula

$$\int_0^2 x f(x)\,dx \approx Af(0) + Bf(1) + Cf(2)$$

exact for all polynomials of degree as high as possible. What is the maximum degree?

Problem 18. We will examine the *Midpoint Rule* in this problem.

(a) Derive the approximation $\int_{-1}^{1} f(x)\,dx \approx Af(0)$ that is exact for all linear polynomials.

(b) Transform the formula in part (a) to one for integration over $[a, b]$.

(c) Transform the formula in part (a) to one for integration over $[x_{i-1}, x_{i+1}]$.

(d) Using part (c), derive the *Composite Midpoint Rule* for approximating $\int_{x_0}^{x_n} f(x)\,dx$. (Assume the nodes $x_0, \ldots, x_n$ are given.)

Problem 19. We will examine the *Trapezoid Rule* in this problem. Suppose the nodes $x_0, \ldots, x_n$ are given.

(a) Using the nodes x_{i-1} and x_i, derive an approximation to $\int_{x_{i-1}}^{x_i} f(x)\,dx$ that is exact for all polynomials of degree ≤ 1.

(b) Determine the error for the quadrature rule in part (a) by integrating the error term for polynomial that interpolates $f(x)$ at x_{i-1} and x_i.

(c) Using part (a), derive the *Composite Trapezoid Rule* for approximating $\int_{x_0}^{x_n} f(x)\,dx$.

(d) Using part (b), derive the error term for the quadrature rule in part (c).

(e) Suppose we use the Composite Trapezoid Rule to approximate $\int_a^b f(x)\,dx$, where $|f''(x)| \leq M$. What is the minimum number of subintervals needed to have the absolute error $< \epsilon$?

Problem 20. Let $f(x)$ be a given function and let $x_0 < x_1 < x_2 < x_3$ be given nodes.

(a) Find the Lagrange form of the polynomial interpolating $f(x)$ at x_1 and x_2.

(b) Derive a quadrature rule to approximate $\int_{x_0}^{x_3} f(x)\,dx$ that uses the nodes x_1 and x_2.

Problem 21. (Math 274 students) Consider the quadrature rule $\sum_{i=1}^{n} A_i f(x_i)$. In this problem, we will examine *Gaussian quadrature* formulas. For Gaussian quadrature, both the coefficients A_i and the nodes x_i need to be determined (this yields methods of higher order than standard Newton-Cotes formulas).

(a) Determine A and x_0 so that the approximation $\int_0^1 f(x)\sqrt{x}\,dx \approx Af(x_0)$ is exact for all linear polynomials.

(b) Determine w and α so that the approximation $\int_{-1}^{1} f(x)\,dx \approx wf(-\alpha)+wf(\alpha)$ is exact for all polynomials of degree ≤ 3.

Problem 22. Suppose $x_0, \ldots, x_n$ are nodes in $[-1,1]$. For a given function f, let p be the degree n polynomial that interpolates f at these nodes. Assume $|f^{(n+1)}(x)| \leq M$ on $[-1,1]$, where $M \in \mathbb{R}$.

(a) If no other information is given about the location of the nodes, what is an upper bound for the error $|\int_{-1}^{1} f(x)\,dx - \int_{-1}^{1} p(x)\,dx|$?

(b) Explain how to choose the $n+1$ nodes so that the approximation is optimized (i.e. the error is minimized).

(c) If we use the nodes in part (b), what is the best upper bound that can be achieved?