Wavelets & Multiresolution Analysis

(A gentle introduction)

Recall that with, e.g., the DFT, we were "analyzing" the signal/data by taking inner products with complex exponentials

$$\hat{x}(k) = \langle x, \varphi^{(k)} \rangle$$

$x \in \mathbb{C}^N$, $\quad \varphi^{(k)} = \left(e^{2\pi i n k/N}\right)_{n=0}^{N-1}$

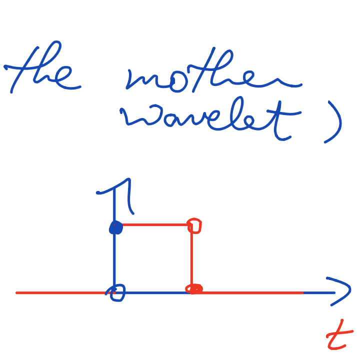
$\hat{x}(k)$ → Fourier coef. corresponding to k^{th} frequency.

With wavelets $\varphi^{(k)}$ will be replaced by dilations & translations of a "mother wavelet"

Let's start with Haar wavelets

Haar Scaling Function: (NOT the mother wavelet)

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$



The subspace V_0 := space of functions of the form

$$f(t) = \sum_{k\in\mathbb{Z}} a_k \phi(t-k), \quad a_k \in \mathbb{R}$$

= piecewise const. functions with possible discont. on $\mathbb{Z}$.

The subspace V_1 := space of functions of the form

$$f(t) = \sum_{k\in\mathbb{Z}} a_k \underbrace{\phi(2t-k)}_{=\phi(2(t-k/2))}, \quad a_k \in \mathbb{R}$$

= piecewise const. functions with possible discont. on $\mathbb{Z}/2$ (half integer)

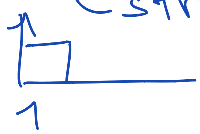
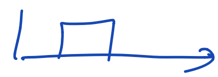
$$V_j := \operatorname{span}\{\phi(2^j t - k),\ k\in\mathbb{Z}\}$$

Observations:

(1) $\forall (j,k) \in \mathbb{Z}^2 : f(2^j t) \in V_j \Rightarrow f(2^j t - k) \in V_j$

(2) $\cdots \subset V_0 \subset V_1 \subset V_2 \cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots$

strict

$\phi(x)$  $\phi(x-1)$ 

$\phi(2x)$  $\phi(2x-1)$ 

$\phi\ ''(2(x-\frac{1}{2}))$

(3) $f(t) \in V_j \Longleftrightarrow f(2t) \in V_{j+1}$

(4) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$ ← $f(x) = 0$

(5) $\bigcup_{j=-\infty}^{\infty} V_j = L^2(\mathbb{R})$

(6) $\exists\, \theta \in V_0$ s.t. $\{\theta(t-k)\}_{k\in\mathbb{Z}}$ $=: \theta_k$ forms a Riesz basis for V_0

Def'n: $\{\theta_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_0 if it is the image of an orthonormal basis for V_0 under an invertible linear transformation

Definition: A sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ is called a Multi-resolution approximation (MRA) if the above 6 properties are satisfied.

HW: Prove that the subspaces $\{V_j\}$ generated by Haar are an MRA.

Two more observations:

- $$\phi(t) = \frac{1}{\sqrt{2}}\left(\sqrt{2}\,\phi(2t)\right) + \frac{1}{\sqrt{2}}\left(\sqrt{2}\,\phi(2t-1)\right)$$
  $$= \underbrace{\sum_k h_\phi(k)\sqrt{2}\,\phi(2t-k)}$$

  For general wavelets, this is the "refinement equation"

- $$\langle \phi_{j,k}, \phi_{j,k'}\rangle = \int_{-\infty}^{\infty} \phi(2^j t - k)\,\phi(2^j t - k')\,dt$$
  $$= \begin{cases} 0 & k \neq k' \\ 1 & k = k' \end{cases}$$

  $2^{j/2}\phi(2^j t - k)$

Haar: $h_\phi(0) = \frac{1}{\sqrt{2}}$, $h_\phi(1) = \frac{1}{\sqrt{2}}$

(rest are 0)

Daub 4 wavelet:

$$h_\phi = \frac{1}{\sqrt{2}}\left(\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right)$$

Back to $V_0 \subset V_1 \subset V_2 \cdots$

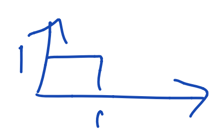
Define W_0: $V_0 \oplus W_0 = V_1$

W_1: $V_1 \oplus W_1 = V_2$

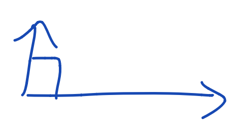
$= V_0 \oplus W_0$

direct sum

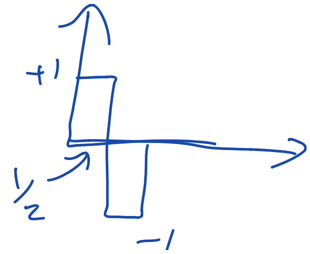
Haar: V_0 spanned by translates of



V_1



$\Rightarrow$ W_0 spanned by translates of



Define $\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \end{cases}$

mother wavelet

Define $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$

$W_j = \text{Span}\{\psi(2^j t - k), k \in \mathbb{Z}\}$

Facts

- $L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots$

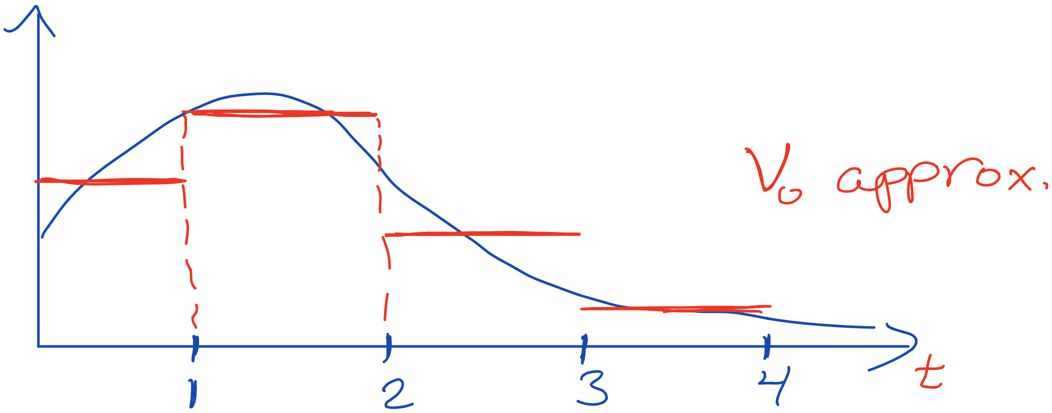
  $\hookrightarrow \{f : \int |f(x)|^2 dx < \infty\}$

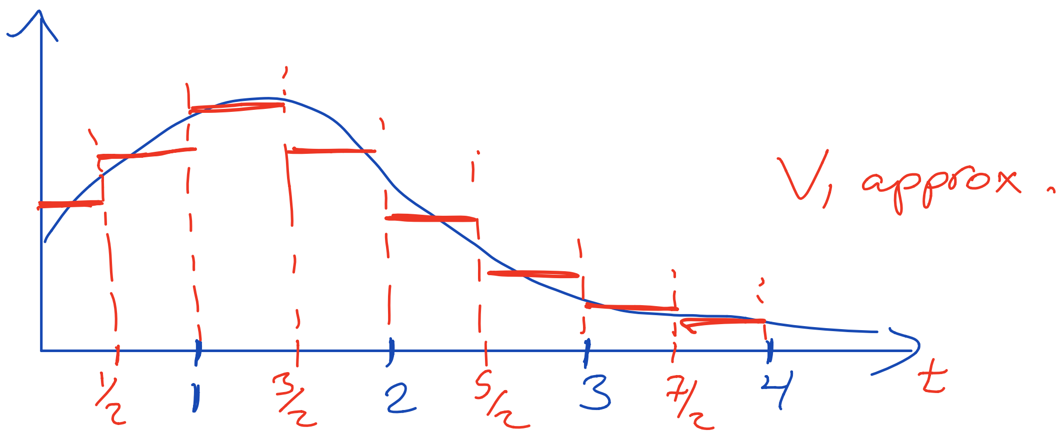
- We can now decompose a function into its components in $V_0, W_0, W_1, \ldots$

- The haar system forms an ONB

  $\{\phi_0(\cdot - k), \psi_{j,k}\}$

  Proof = Exercise





$\vdots$

The Discrete Wavelet Transform

- In practice, there is always a finest scale and a coarsest scale that we care about. (So we are satisfied with approx. in V_j) finest scale
- Assume the finest scale is $2^0 = 1$
- So $f(t) \rightsquigarrow (x_0, x_1, x_2, \dots x_{N-1})$

where $x_k = \langle f, \varphi_{J,k} \rangle$

finest scale

$x \in \mathbb{R}^N$ is now our signal!

and we are satisfied with the approx

$$f \approx \underbrace{\sum_{k=0}^{N-1} x_k \, \phi_{J,k}}_{f_J} \qquad (\text{Proj of } f \text{ onto } V_J)$$

Henceforth, we will work in this setting

Define the two operators

"wavelet"

$$H(x)_k = (h * x)_k = \frac{1}{2} x_k - \frac{1}{2} x_{k+1}$$

$$h = (\ldots 0, \ldots, -\tfrac{1}{2}, \tfrac{1}{2}, 0 \ldots)$$

$k = \frac{-1}{2}$ $\qquad$ $k = 1$

"scaling"

$$L(x)_k = (\ell * x)_k = \frac{1}{2} x_k + \frac{1}{2} x_{k+1}$$

$$\ell = (0, \ldots, \tfrac{1}{2}, \tfrac{1}{2}, 0 \ldots)$$

Now, keep only the even subscripts on $H(x)$ & $L(x)$:

$$[DH(x)]_k = [H(x)]_{2k} = \frac{1}{2}x_{2k} - \frac{1}{2}x_{2k+1}$$

↳ Downsampling

$$[DL(x)]_x = [L(x)]_{2k} = \frac{1}{2}x_{2k} + \frac{1}{2}x_{2k+1}$$

Idea: Recall $f_J = \sum_k x_k \phi_{J,k}$

we can call $(x_0, ----, x_{N-1}) =: a^{(J)}$

so that $f_J = \sum_k a_k^{(J)} \phi_{J,k}$

and since $V_J = V_{J-1} \oplus W_{J-1}$

then $f_J = \sum_k a_k^{(J-1)} \underbrace{\phi_{J-1,k}}_{\phi(2^{J-1}x-k)} + \sum_k b_k^{(J-1)} \psi_{J-1,k}$

(no normalization)

where $\phi(x) = \phi(2x) + \phi(2x-1)$

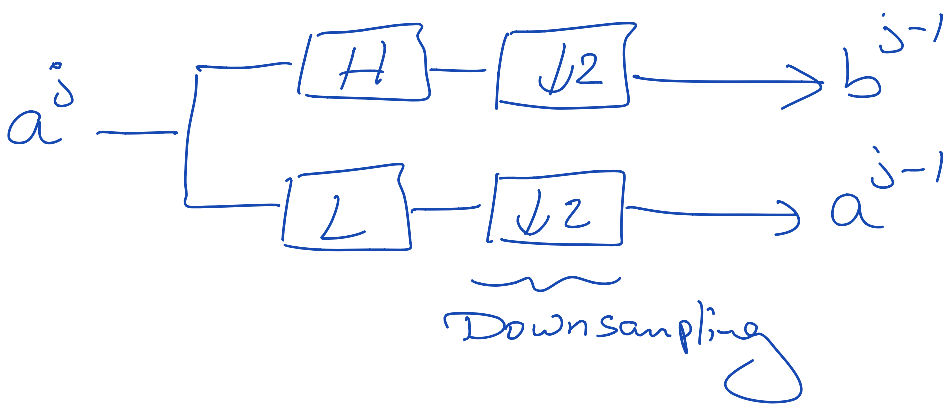
$\psi(x) = \phi(2x) - \phi(2x-1)$

$$\Rightarrow \quad \phi(2^{J-1}x) = \phi(2^J x) + \phi(2^J x - 1)$$
$$\psi(2^{J-1}x) = \phi(2^J x) - \phi(2^J x - 1)$$

$$\Rightarrow \quad \ldots \quad \Rightarrow \begin{cases} a_k^{J-1} = \dfrac{a_{2k}^J + a_{2k+1}^J}{2} \\[2ex] b_k^{J-1} = \dfrac{a_{2k}^J - a_{2k}^J}{2} \end{cases}$$

We can proceed in this way to get $f_J = \underbrace{f_0 + w_0 + w_1 + \cdots}_{f_{J-1}} + w_{J-1}$

Pictorially, at each level j



So now, if x is of length $N = 2^n$ we will end up with

b^{J-1} of length $N/2 = 2^{n-1}$

b^{J-2} " " 2^{n-1}

b^1 of length 2

b^0 of length 1

a^0 of length 1

These coefficients are the DWT coefficients of x

Remark: Could have stopped at any a^k, $k \geq 0$, $k \leq J$

(# of "levels" in wavelet transform)

What about Reconstruction?

$\Leftrightarrow$ What about IDWT?

Since the DWT is invertible, we just need to figure out how to combine the coef's to get x back.

(why?)

Turns out (check!) that

$$a^j = \tilde{L} U a^{j-1} + \tilde{H} U b^{j-1}$$

upsampling

$$Uz = (\dots, 0, z_0, 0, z_1, 0, z_2, 0, \dots)$$

$$(\tilde{L} z) = \tilde{\ell} * z$$

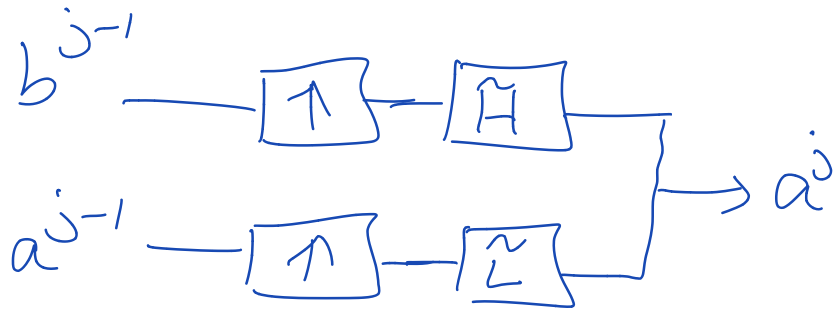
$$(\dots, 0, \dots, 0, \underbrace{1, 1}_{k=0,1}, 0 \dots)$$

$$(\tilde{H} z) = \tilde{h} * z$$

$$(\dots 0, \dots, \underbrace{1, -1}_{k=0,1}, 0, \dots)$$

$k=0))$

$\Longrightarrow$



Note: We have barely touched the surface of wavelet theory.

Theorem: Let ϕ be continuous with compact support and suppose

$$\int \phi(t-k)\,\phi(t-\ell)\,dt = \delta_{k,\ell}.$$

Let $V_j = \text{span}\{\phi(2^j x - k),\ k \in \mathbb{Z}\}$

Then: (1) $\displaystyle\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$

(2) If $\begin{cases} \int \phi(x)\,dx = 1 \\ \phi(x) = \sum_k p_k\,\phi(2x-k) \end{cases}$

for some finite # of P_k's

then $\displaystyle\bigcup_{j=-\infty}^{\infty} V_j = L^2(\mathbb{R})$

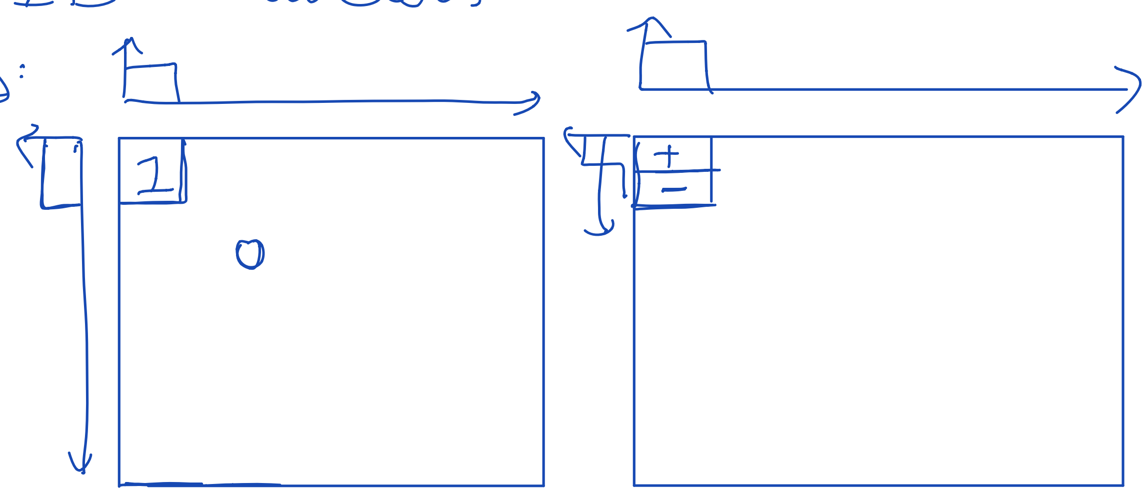
In particular $\{V_j\}$ forms an MRA

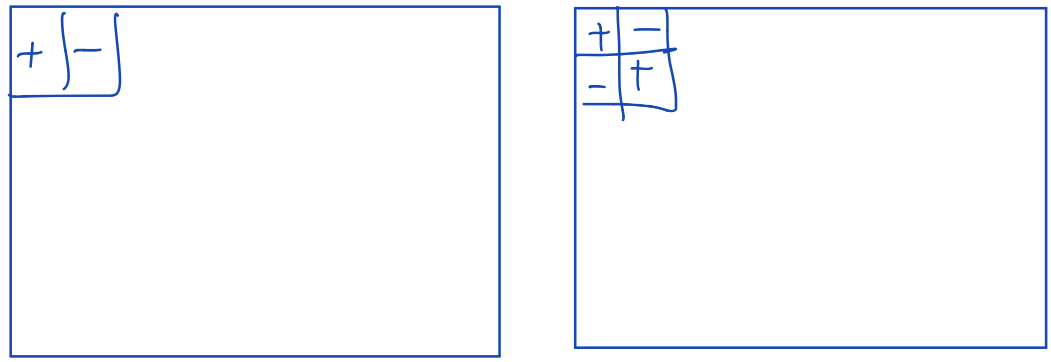
Proof: See Theorem 5.3 in Boggess Book.

—x—

2D Wavelets ≈ outer product 1D wavelets

e.g.:





and so on

x

Applications (examples)

Compression: Given an natural image $X \in \mathbb{R}^{N\times N}$ and its vectorized version $x \in \mathbb{R}^{N^2}$, we can compute its wavelet coeff.

$y = Wx$, y will tend to be sparse i.e. most entries of y will be ≈ 0.

(JPEG 2000, FBI fingerprint database)

Denoising: Suppose

$$y = x + n \in \mathbb{R}^N$$

signal   noise

and let, say, $n \sim \mathcal{N}(0, I)$

then $W y = \underbrace{W x}_{\text{sparse}} + \underbrace{W n}_{\text{Gaussian}}$

Shrinkage: keep only the large coeffs of Wy (mostly)