Homework 2 for Math 277A - Fall 2018

Solve 3 of the following 4 problems.

(I) Given $A \in \mathbb{R}^{m\times N}$, prove that every non-negative s-sparse vector $x \in \mathbb{R}^N$ is the unique solution to

$$\min_{z\in\mathbb{R}^N} \|z\|_1 \text{ subject to } Ax = Az,\ z \geq 0$$

if and only if

$$v_{S^c} \geq 0 \implies \sum_{j=1}^{N} v_j > 0$$

for all $v \in \text{null}(A) \setminus \{0\}$ and all $S \subset [N]$ with $|S| \leq s$.

(II) Recall the definition of the ℓ_1 coherence function

$$\mu_1(s) := \max_{i\in[N]} \max \left\{ \sum_{j\in S} |\langle a_i, a_j\rangle|, S \subset [N], |S| = s, i \notin S \right\},$$

and the definition of the $\|\cdot\|_{1\to 1}$ norm (of a matrix):

$$\|M\|_{1\to 1} := \max_{z:\|z\|_1\leq 1} \|Mz\|_1$$

(a) Prove that $\mu_1(s) = \max_{S:|S|\leq s+1} \|{A_S}^* A_S - I\|_{1\to 1}$.

(b) Compare the restricted isometry property constant δ_s to $\mu_1(s)$ (for the same matrix of course).

(III) Let $A \in \mathbb{R}^{m\times N}$ be a matrix with RIP constant δ_s. Prove that if $x \in \mathbb{R}^N$,

$$\|Ax\|_2 \leq (1+\delta_s)^{1/2} \left(\|x\|_2 + \frac{\|x\|_1}{\sqrt{s}} \right)$$

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(IV) (Bernoulli Selector) Let $U \in \mathbb{C}^{N\times N}$ be a unitary matrix with BOS constant K and let ε_j be random variables that take the value 1 with probability m/N and 0 with probability $1 - m/N$. Define the *random sampling set*

$$T := \{j \in [N] \quad | \quad \varepsilon_j = 1\},$$

and let A be the random submatrix of U consisting of the rows indexed by T.

(a) Show that $\mathbb{E}|T| = m$,

(b) Find an upper bound for $\mathbb{P}(\big||T| - m\big| \geq t)$ for $t > 0$.

(c) Let $S \subset [N]$ with $|S| = s$, and let $\widetilde{A} = \sqrt{N/m}A$. Verify that

$$\widetilde{A}^* \widetilde{A} = \frac{N}{m} \sum_{j=1}^{N} \varepsilon_j X_j X_j^*$$

where $(X_j)_\ell = \bar{U}_{\ell,j}$ and use the matrix Bernstein inequality to derive an upper bound for

$$\mathbb{P}(\|\widetilde{A}_S^* \widetilde{A}_S - I\|_{2\to 2} \geq t)$$

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