Solve 3 of the following 4 problems.

(I) Given $A \in \mathbb{R}^{m \times N}$, prove that every non-negative $s$-sparse vector $x \in \mathbb{R}^N$ is the unique solution to

$$\min_{z \in \mathbb{R}^N} \|z\|_1 \text{ subject to } Ax = Az, \ z \geq 0$$

if and only if

$$v_S \geq 0 \implies \sum_{j=1}^N v_j > 0$$

for all $v \in \text{null}(A) \setminus \{0\}$ and all $S \subset [N]$ with $|S| \leq s$.

(II) Recall the definition of the $\ell_1$ coherence function

$$\mu_1(s) := \max_{i \in [N]} \max_{S \subset [N], |S| = s, i / \notin S} \left\{ \sum_{j \in S} |\langle a_i, a_j \rangle| : \right\}$$

and the definition of the $\| \cdot \|_1$ norm (of a matrix):

$$\|M\|_1 := \max_{z : \|z\|_1 \leq 1} \|Mz\|_1$$

(a) Prove that $\mu_1(s) = \max_{|S| \leq s+1} \|A_S^*A_S - I\|_{1 \to 1}$.

(b) Compare the restricted isometry property constant $\delta_s$ to $\mu_1(s)$ (for the same matrix of course).

(III) Let $A \in \mathbb{R}^{m \times N}$ be a matrix with RIP constant $\delta_s$. Prove that if $x \in \mathbb{R}^N$,

$$\|Ax\|_2 \leq (1 + \delta_s)^{1/2} \left( \|x\|_2 + \frac{\|x\|_1}{\sqrt{s}} \right)$$

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(IV) (Bernoulli Selector) Let $U \in \mathbb{C}^{N \times N}$ be a unitary matrix with BOS constant $K$ and let $\varepsilon_j$ be random variables that take the value 1 with probability $m/N$ and 0 with probability $1 - m/N$. Define the random sampling set

$$T := \{ j \in [N] \mid \varepsilon_j = 1 \}$$

and let $A$ be the random submatrix of $U$ consisting of the rows indexed by $T$.

(a) Show that $\mathbb{E}[T] = m$,

(b) Find an upper bound for $\mathbb{P}(|T| - m) \geq t$ for $t > 0$. 

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(c) Let $S \subset [N]$ with $|S| = s$, and let $\bar{A} = \sqrt{N/m}A$. Verify that

$$\bar{A}^* \bar{A} = \frac{N}{m} \sum_{j=1}^{N} \varepsilon_j X_j X_j^*$$

where $(X_j)_{\ell} = \bar{U}_{\ell,j}$ and use the matrix Bernstein inequality to derive an upper bound for

$$\mathbb{P}(\|\bar{A}^*_S \bar{A}_S - I\|_{2 \rightarrow 2} \geq t)$$