6.1 Chebyshev Polynomials

Recall that the error in polynomial interp is given by

\[ f(x) - p(x) = \frac{1}{(n+1)!} \sum_{i=0}^{n} f^{(n+1)}(x_i) \prod_{j=0}^{n} (x-x_j) \]

Assume (for convenience) that the interpolation nodes are in \([-1,1]\). If \( x \in [-1,1] \) then \( f(x) \in [-1,1] \), so

\[ \max_{x \in [-1,1]} |f(x) - p(x)| \leq \frac{1}{(n+1)!} \max_{x \in [-1,1]} |f^{(n+1)}(x)| \cdot \max_{x \in [-1,1]} \prod_{i=0}^{n} |x-x_i| \]

Idea: choose the nodes \( x_i \) to minimize this term!
Observe: \( \prod_{i=0}^{n} (x-x_i) \) is a monic polynomial with coefficient of \( x^n \) is 1.

**Theorem:** If \( P \) is a monic polynomial of degree \( n \) then

\[
\|P\|_\infty := \max_{x \in [-1,1]} |P(x)| \geq 2^{-n}
\]

To prove this, we will construct a polynomial that achieves the bound.

**Chebyshev Polynomials:**

2 Definitions (equivalent)

**Recursive Definition:**

\[
\begin{cases}
T_0(x) = 1, & T_1(x) = x \\
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), & n \geq 1
\end{cases}
\]
\[ T_2(x) = 2x^2 - 1 \]
\[ T_3(x) = 4x^3 - 3x \]
\[ T_4(x) = 8x^4 - 8x^2 + 1 \quad \ldots \]

**Equivalent Definition/Theorem**

\[ T_n(x) = \cos(n \cos^{-1} x), \quad n \geq 0 \]

**Proof of equivalence:**

Since \( \cos(A + B) = \cos A \cos B - \sin A \sin B \)

\[ \Rightarrow \cos(n+1)\theta = \cos \theta \cos n\theta - \sin \theta \sin n\theta \]
\[ \cos(n-1)\theta = \cos \theta \cos n\theta + \sin \theta \sin n\theta \]

\[ \Rightarrow \cos(n+1)\theta + \cos(n-1)\theta = 2\cos \theta \cos n\theta \]

Plug in \( \cos^{-1} x \)

Let define \( f_n(x) = \cos(n \cos^{-1} x) \rightarrow f_0(x) = 1, f_1(x) = x \)
\[ \& \quad f_{n+1}(x) + f_{n-1}(x) = 2x f_n(x) \]
\[ s_{n+1}(x) = 2x \hat{s}_n(x) - s_n(x) \]

So \( T_n = s_n \quad \forall n \)

Recall that we want to get an upper bound on \( |p(x)|, \ x \in [-1, 1] \) when \( p \) is monic.

**Theorem:** \( \|p\|_\infty = \max_{x \in [-1, 1]} |p(x)| \geq 2^{1-n} \)

**Proof:** Suppose that \( |p(x)| < 2^{1-n} \quad \forall x \in [-1, 1] \)

We want to get a contradiction.

Let \( q^i = 2^{-n} T_n(x) \) & let \( x_i = \cos\left(\frac{i \pi}{n}\right) \).

Observe that \( q \) is monic, degree \( n \). Also:

\[ q(x_i) = 2^{-n} \cos\left(n \cdot \frac{i \pi}{n}\right) = 2^{-n} (-1)^i \]

So \( q(x_i)(-1)^i = 2^{-n} \geq |p(x_i)| = (-1)^i p(x_i) \)

by our supposition

\[ (-1)^i (q(x_i) - p(x_i)) > 0 \quad \forall i \in \{0, \ldots, n\} \]

poly, degree \( n-1 \) (highest degree terms cancel)
So, we have a polynomial of degree \( \leq n-1 \) that changes sign \( n+1 \) times in \([-1,1] \) \( \Rightarrow \) it has \( n \)-roots. Can't happen with degree \( \leq n-1 \) \( \Rightarrow \) contradiction \( \Rightarrow |P_n(x)| \geq 2^{1-n} \)

OK, let's get back to the error in poly interp.

\[
\max_{x \in [-1,1]} |f(x) - P_n(x)| \leq \frac{1}{(n+1)!} \max_{x \in [-1,1]} |(n+1)(x)| \max_{x \in [-1,1]} \left| \prod_{i=0}^{n} (x-x_i) \right|
\]

Monic degree \( n \) \( \Rightarrow \geq 2^{-n} \).

So, the best we can do is \( 2^{-n} \).

From the proof above, we want \( \frac{n}{i=0} \prod (x-x_i) \)

With nodes \( x_i = \cos \left( \frac{2i+1}{2n+2} \pi \right), \quad i = 0, -1, \ldots, n \)
If $x_i$ are the the roots of $T_{n+1}$:

$$|f(x) - p(x)| \leq \frac{2^{-n}}{(n+1)!} \max_{|t| \leq 1} |f^{(n+1)}(t)|$$