6.1 Polynomial Interpolation

Goal: Given a set of \( n+1 \) data points \((x_i, y_i)\)

![Diagram showing data points \(x_0, x_1, \ldots, x_n\) and \(y_0, y_1, \ldots, y_n\)]

We seek a polynomial \( P \) with lowest possible degree so that \( P(x_i) = y_i \).

We say that \( P \) interpolates the data.

**Theorem:** If \( x_0, x_1, \ldots, x_n \) are distinct real numbers, then for arbitrary \( y_0, y_1, \ldots, y_n \), there exists a unique polynomial \( P_n \) of degree \( n \) or less such that

\[
P_n(x_i) = y_i \quad \text{for} \quad i = 0, 1, \ldots, n
\]

**Proof:** (1) **Existence:**

We'll do it by induction.

- For \( n = 0 \), we can always find \( P_0 \) s.t. \( P_0(x_0) = y_0 \).

- Suppose \( P_{k-1} \) satisfies \( P_{k-1}(x_i) = y_i \) for \( i = 0, 1, \ldots, k-1 \).

Let \( P_k(x) = P_{k-1}(x) + C(x-x_0)(x-x_1)\ldots(x-x_{k-1}) \)

To find \( C \) we just solve \( P_k(x_k) = y_k \).

So \( P_k(x_k) = P_{k-1}(x_k) + C(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1}) \)

\( \Rightarrow \) can solve!

\( y_k \) known because \( P_{k-1} \) is known (all are \( x_i \)'s distinct)
We also note that $P_k(x_i) = P_{k-1}(x_i) = y_i$ for $i = 0, \ldots, k$.

So $P_k$ interpolates the data $(x_i, y_i)$, $i = 0, \ldots, k$.

Uniqueness: Suppose $\exists q_n$ that interpolates the data

then $q_n(x_i) - P_n(x_i) = 0$ for $i = 0, \ldots, n$

$\Rightarrow (q_n - P_n)(x_i) = 0 \rightarrow n + 1$ of these

... of degree $\leq n$ with $n + 1$ zeros $\Rightarrow q_n - P_n = 0$.

Newton form of the interpolating polynomial

From the proof: $P_k(x) = P_{k-1}(x) + C_k(x-x_0) \cdots (x-x_0)$

$\hspace{1cm} = \cdots$

$\hspace{1cm} = C_0 + C_1(x-x_0) + C_2(x-x_0)(x-x_2) + \ldots$

$\hspace{4cm} + C_k(x-x_0) \cdots (x-x_{k-1})$

In short form: $P_k(x) = \sum_{i=0}^{k-1} C_i \frac{x-x_0}{1!} \frac{x-x_1}{2!} \cdots \frac{x-x_i}{i!}$ (interpolating polynomials in Newton form).
Lagrange form of interpolating polynomial

We can write our polynomial as

\[ P_n(x) = \sum_{k=0}^{n} y_k \ell_k(x) \]

degree on polynomials depending on the nodes \( x_0, x_1, \ldots, x_n \)

may choose specifically we take \( \ell_k(x_i) = \delta_{ik} \)

\[ \ell_k(x) = \delta_{ik} \]

this can be done, e.g., by setting

\[ \ell_0(x) = c \prod_{i=1}^{n} \frac{1}{(x-x_i)} \Rightarrow c = \frac{\ell_0(x_0)}{\prod_{i=1}^{n} (x-x_i)} = \frac{1}{\prod_{i=1}^{n} (x_0-x_i)} \\
\]

\[ \Rightarrow \ell_0(x) = \prod_{i=1}^{n} \frac{x-x_i}{x_0-x_i} \]

Similarly

\[ \ell_0(x) = \prod_{i \neq 0}^{n} \frac{(x-x_i)}{(x_0-x_i)} \]

\[ P_n(x) = \sum_{i=0}^{n} y_i \ell_i(x) \]
Example: Given \[ x | 5 \quad -7 \quad -6 \quad 0 \]
\[ y | 1 \quad -23 \quad -54 \quad 954 \]

Find the cardinal functions and the Lagrange form of the interpolating polynomial

\[ l_0(x) = \frac{x - (-7)}{5 - (-7)} \cdot \frac{x - (-6)}{5 - (-6)} \cdot \frac{x - 0}{5 - 0} = \frac{(x+6)(x+7)x}{660} \]

\[ l_1(x) = \frac{x - -5}{7 - -7} \cdot \frac{x - -6}{7 - -6} \cdot \frac{x - 0}{7 - 0} = \frac{(x-5)(x+6)x}{-84} \]

\[ l_2(x) = \quad \]
\[ l_3(x) = \quad \]

\[ \Rightarrow P_3(x) = l_0(x) - 23 l_1(x) - 54 l_2(x) - 954 l_3(x) \]

**Theorem (error in polynomial interp)**

Let \( f \in C^{n+1}([a,b]) \) & \( P \) be a poly of degree \( \leq n \)

that interpolates \( f \) at \( n+1 \) distinct pts \( x_0, x_1, \ldots, x_n \in [a,b] \).

Then \( \forall x \in [a,b], \exists \delta_x \in (a,b) \) s.t.

\[ f(x) - P(x) = \frac{1}{(n+1)!} \sum_{i=0}^{n} f^{(i)}(\delta_x) \frac{x^{n-i}(x-x_i)}{i!} \]

Exercise: read the proof! HW: 6.1: 4(a,b), 8, 14, 22
6.2 Divided Differences

Recall:

\[ P_n(x) = \sum_{j=0}^{n} c_j \frac{(x-x_j)}{\prod_{k=j}^{n} (x-x_k)} \]

interp. polynomials in Newton form.

If we call

\[ g_0(x) = 1 \]
\[ g_1(x) = x-x_0 \]
\[ g_2(x) = (x-x_0)(x-x_1) \]
\[ \vdots \]
\[ g_n(x) = \prod_{i=0}^{n-1} (x-x_i) \]

then \[ P_n(x) = \sum_{j=0}^{n} c_j g_j(x) \]

Recall, the \( c_j \)'s are what we need to find.

We have \( n+1 \) linear equations:

\[ \sum_{i=0}^{n} c_i g_j(x_i) = f(x_i), \quad i = 0, 1, \ldots, n+1 \]
In matrix form:

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & x_1 - x_0 & 0 & \cdots & 0 \\
1 & x_2 - x_0 & (x_2 - x_0)(x_1 - x_0) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n - x_0 & (x_n - x_0)(x_{n-1} - x_0) & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}
= \begin{bmatrix}
y_0 \\
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
\]

So: \( c_0 \) depends on \( y_0 \), we write \( c_0 = f[x_0] \)

\( c_1 \) depends on \( y_0, y_1 \), we write \( c_1 = f[x_0, x_1] \)

\( \vdots \)

\( c_n \) depends on \( y_0, y_1, \ldots, y_n \), we write \( c_n = f[x_0, x_1, \ldots, x_n] \)

\[ \text{old notation} \]

Let's compute a couple of these:

- \( f[x_0] = f(x_0) \) \( (= c_0) \)
- \( f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \) \( (= c_1) \)

& \( P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \)

\[ \text{Theorem: } f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} \]

Proof: exercise / read it.
So \( f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \)

This makes it easy to compute the \( f[x_i] \)'s using a divided differences table

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( f[x_0] )</th>
<th>( f[x_0, x_1] )</th>
<th>( f[x_0, x_1, x_2] )</th>
<th>( f[x_0, x_1, x_2, x_3] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( f[x_1] )</td>
<td>( f[x_1, x_2] )</td>
<td>( f[x_1, x_2, x_3] )</td>
<td></td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( f[x_2] )</td>
<td>( f[x_2, x_3] )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_3 )</td>
<td>( f[x_3] )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

E.g.: Use divided differences to find the Newton interpolating polynomial for

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Sol'n: Step 1 (table)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f[x] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( \frac{-3-1}{1-3} = 2 )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{2-(-3)}{5-3} = \frac{5}{2} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{4-2}{6-5} = 2 )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{3\frac{7}{8}-\frac{7}{8}}{6-3} = \frac{7}{40} )</td>
</tr>
</tbody>
</table>
Step 2: write the polynomial

\[ P(x) = 1 + 2(x-3) - \frac{3}{8}(x-3)(x-1) \]
\[ + \frac{2}{40} (x-3)(x-1)(x-5) \]

Property: If \((z_0, \ldots, z_n)\) is a permutation of \((x_0, \ldots, x_n)\) then

\[ f[z_0, \ldots, z_n] = f[x_0, \ldots, x_n] \]

both are the ways of \(x^n\) in the interp. poly

HW 6.2: 4, 8, 9, 24

6.3 Hermite Interpolation

Want to interpolate not only the function, but also its derivatives

Setup: at each \(x_i\) we are given \(f^{(j)}(x_i)\) for \(0 \leq j \leq k_i - 1\)

e.g. Find a polynomial \(P\) with \(P(0) = 0, P'(0) = 1 \& P(1) = 1\)
In general:

\[ p^{(i)}(x_i) = c_{ij} \quad (0 \leq j \leq k_i - 1, \ 0 < i \leq n) \]

where \( \sum_{i=0}^{n} k_i = m + 1 \)

So, we have \( m+1 \) conditions \( \Rightarrow \) reasonable to look for an \( m \)th degree polynomial.

**Theorem:** There is a unique polynomial of degree at most \( m \) satisfying

\[ p^{(i)}(x_i) = c_{ij} \quad (0 \leq j \leq k_i - 1, \ 0 < i \leq n) \]

where \( \sum_{i=0}^{n} k_i = m + 1 \)

**Example:** Hermite interpolation with only one node.

Given: \( p^{(i)}(x_0) \) for \( 0 \leq i \leq k \)

\[ p(x) = p(x_0) + p'(x_0)(x-x_0) + p''(x_0)(x-x_0)^2/2! \]

\[ + \ldots + p^{(k)}(x_0)(x-x_0)^k/k! \]

*Taylor polynomial!*
Newton Divided Difference Method

Extensions to some Hermite interpolation

Example: Use extended Newton divided difference to find a polynomial satisfying

\[ P(0) = 2, \ P'(0) = 3, \ P(2) = 6, \ P'(2) = 7, \ P''(2) = 8 \]

will come back to it.

Method:

\[
\begin{array}{c|ccccc}
 & x_0 & x_1 & x_2 & x_3 & x_4 \\
\hline
x_0 & f[x_0] & f[x_0,x_0] & f[x_0,x_0,x_1] & f[x_0,x_0,x_1,x_2] & f[x_0,x_0,x_1,x_2,x_3] \\
x_0 & f[x_0] & f[x_0,x_1] & f[x_0,x_1,x_2] & f[x_0,x_1,x_2,x_3] \\
x_1 & f[x_1] & f[x_1,x_2] & f[x_1,x_1,x_3] \\
x_2 & f[x_2] \\
\end{array}
\]

But what is \( f[x_0,x_0] \)? \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \)

\[ f[x_0,x_0,x_0] = ? \]

Apply Theorem 4: \( \exists \ \delta = x_0 : f[x_0,x_0,x_0] = \frac{f'(x_0)}{2} \)
Similarly \[ f[x_0, \ldots, x_0] = \frac{f^{(k)}(x_0)}{k!} \] for all time.

OK back to the example.

\[ P(1) = 2, \quad P'(1) = 3, \quad P(2) = 6, \quad P'(2) = 7, \quad P''(2) = 8 \]

\[
\begin{array}{c|cccc}
1 & 2 & 3 & y_{3-1} & 2 \\
1 & 2 & 5 & \frac{y_{2-1}}{2} & -1 \\
2 & 6 & 7 & \frac{8}{2!} & \\
2 & 6 & & & \\
\end{array}
\]

Now we use the top row to write \( p(x) \)

\[ p(x) = 2 + 3(x-1) + 1(x-1)^2 + 2(x-1)^2(x-2) - (x-1)^3(x-2) \]

Check: \[ p(1) = 2 \]
\[ p'(1) = 3 \]
\[ p(2) = 2 + 3(2-1) + 1(2-1)^2 + 0 = 6 \]
\[ p''(2) = 3 + 2(2-1) + 2(2-1)^2 = 7 \]
\[ p''(2) = 2 + \ldots = 8 \]

HW 6.3: 1, 3