7.2 Numerical Integration based on Interpolation

Idea: Some functions are easy to integrate analytically, e.g., \( f(x) = e^x \), \( f(x) = \sin x \).

Others, like \( f(x) = e^{x^2} \) or \( g(x) = \int_0^x \sin(x) \, dx \), may not be.

Goal: Want a method to approximate \( \int_a^b f(x) \, dx \) by using only \( f(x_i), i = 0, \ldots, n \).

Reasonable approach: Use polynomial interpolation to write \( f(x) \approx p(x) = \sum_{i=0}^n f(x_i) l_i(x) \).

Then "hope" that \( \int_a^b f(x) \, dx \approx \int_a^b p(x) \, dx \).
So \[ \int_a^b f(x) \, dx = \sum_{i=0}^{n-1} \frac{b-a}{n} f(x_i) e_i(x) \, dx \]

exchange order \[ \int_a^b \]

\[ = \sum_{i=0}^{n-1} f(x_i) \int_a^b e_i(x) \, dx \]

\( f(x_i) \) is a number \[ \sum_{i=0}^{n-1} \] call this \( A_i \)

\[ = \sum_{i=0}^{n-1} A_i f(x_i) \]

In fact, you already know some examples of this

**Trapezoidal rule:** When \( n = 1 \) & \( x_0 = a, x_1 = b \)

we have \( e_0(x) = \frac{x_1-x}{x_1-x_0} = \frac{b-x}{b-a} \)

\( e_1(x) = \frac{x_0-x}{x_0-x_1} = \frac{x-a}{b-a} \)

\[ \Rightarrow A_0 = \int_a^b \frac{b-x}{b-a} \, dx = \frac{b(b-a)}{b-a} - \frac{b^2-a^2}{2(b-a)} = b - \frac{b+a}{2} \]

\( A_0 = \frac{b-a}{2} \)

\( \Rightarrow \int_a^b f(x) \, dx \approx \frac{b-a}{2} [f(a) + f(b)] \)

\( A_1 = \frac{b-a}{2} \) (similarly)
Composite trapezoidal rule:

1. Subdivide \([a, b]\) into \(n\) pieces using \(a = x_0 < x_1 < x_2 < \cdots < x_n < b\).

2. Use the trapezoidal rule for each piece.

\[
\int_a^b f(x) \, dx \approx \frac{1}{2} \sum_{i=1}^{n} (x_i - x_{i-1}) [f(x_i) + f(x_{i-1})]
\]
Comp. Trap. rule with equal spacing

when all the subintervals of \([a,b]\) are the same length \(h = x_i - x_{i-1}\)

\[
\int_a^b f(x) \, dx \approx \frac{h}{2} \sum_{i=1}^{n} \left( f(x_{i-1}) + f(x_i) \right) = \frac{h}{2} \left[ f(a) + 2 \sum_{i=1}^{n-1} f(a+ih) + f(b) \right]
\]

Back to the general, non-composite case

Recall:

\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^{n} A_i f(x_i) \quad \rightarrow \quad \int_a^b f_i(x) \, dx
\]

Since for polynomials of degree \(\leq n\)

\[f(x) = \sum_{i=0}^{n} f(x_i) \, l_i(x)\]

then our integration formula is exact for poly of deg \(\leq n\)
This observation allows us to find the $A_i$'s "easily" by the method of undetermined coefficients.

**Example:** $n = 2$, $[a, b] = [0, 1]$ & $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$

$$\Rightarrow \int_0^1 f(x) \approx A_0 f(0) + A_1 f(1) + A_2 f(2)$$

Formula is exact for poly. of degree $\leq 2$

So, $$\int_0^1 f(x) \, dx = A_0 + A_1 + A_2$$

$$\int_0^{\frac{1}{2}} f(x) \, dx = A_0 \cdot 0 + A_1 \cdot \frac{1}{2} + A_2$$

$$\int_0^{\frac{1}{2}} x \, dx = A_0 \cdot 0 + A_1 \cdot \frac{1}{2} + A_2$$

$$\int_0^{\frac{1}{2}} x^2 \, dx = A_0 \cdot 0 + A_1 \cdot \frac{1}{4} + A_2$$

3 eq'n & 3 unknowns $\Rightarrow A_1 = \frac{2}{3}$, $A_2 = \frac{1}{6} = A_0$

$\Rightarrow \int_0^1 f(x) \, dx \approx \frac{1}{6} f(0) + \frac{2}{3} f(1) + \frac{1}{6} f(1)$
Simpson's Rule

Repeat the same calculation but with arbitrary \([a, b]\) and \(x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b\) to get

\[
\int_a^b f(x) \, dx \approx \frac{b-a}{6} \left( f(a) + 4f(\frac{b+a}{2}) + f(b) \right)
\]

Remark: should be exact for poly. of degree \(\leq 2\)

but is also exact for poly. of degree \(\leq 3\)

Composite Simpson's rule can also be used (see book)

Error Analysis

Want an expression for the error in numerical integration; that is, we want an expression for

\[
\int_a^b f(x) \, dx - \sum_{i=0}^{\infty} A_i f(x_i)
\]
Recall: \( A_i = \int_a^b f_i(x) \, dx \) where \( f_i(x) \) comes from the Lagrange interpolating poly:

\[
P(x) = \sum_{i=0}^{n} f(x_i) L_i(x)
\]

Also: \( f(x) - P(x) = \frac{1}{(n+1)!} \int_a^b \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x-x_i) \, dx \)

So:

\[
\int_a^b f(x) - \sum_{i=0}^{n} A_i f(x_i) = \frac{1}{(n+1)!} \int_a^b \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x-x_i) \, dx
\]

Can't do much about this as it depends on the \( f^{(n+1)} \) we're numerically integrating. Reasonable to just use an upper bound

\[
M = \max_{x \in [a,b]} |f^{(n+1)}(x)|
\]

Therefore

\[
\int_a^b f(x) \, dx - \sum_{i=0}^{n} A_i f(x_i) \leq \frac{M}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x-x_i) \, dx
\]

Previously we used the roots of the Chebyshev Poly as our nodes \( x_i \) to minimize \( \prod_{i=0}^{n} (x-x_i) \). We saw that gives

\[
\prod_{i=0}^{n} (x-x_i) = 2^n
\]

Now, we want to minimize \( \int_a^b \prod_{i=0}^{n} (x-x_i) \, dx \)
Turns out, when $[a, b] = [-1, 1]$, the choice of nodes

$$x_i = \cos \left( \frac{(i+1)\pi}{n+2} \right) \quad i = 0, \ldots, n$$

is optimal. See Theorem 1 on page 487.

These are the roots of the polynomial

$$U_{n+1}(\cos \theta) = \frac{\sin((n+2)\theta)}{\sin \theta}$$

$$\Rightarrow U_{n+1}(x) = \frac{\sin((n+2)\cos^2 \theta)}{\sin(\cos^2 \theta)}$$

Remark: Just as $T_n(x)$ was not monic, $U_{n+1}(x)$ is not monic. However, $2^{-(n+1)}U_{n+1}$ is monic.

So

$$2^{-(n+1)}U_{n+1} = \prod_{i=0}^{n} (x - x_i)$$

where $x_i = \cos \left( \frac{(i+1)\pi}{n+2} \right)$

Moreover,

$$\int_{-1}^{1} \frac{n}{\prod_{i=0}^{n} (x - x_i)} |dx| = 2^{-n}$$
So, with the choice of nodes:

\[
|S_n f(x) - \sum_{i=0}^n A_i f(x_i)| \leq \frac{M}{(n+1)!} 2^n
\]

Remark: Just as with \( T_n(x) \), \( U_n(x) \) can be generated recursively via:

\[U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1} = 2x U_n - U_{n-1} \quad \text{when } n \geq 1\]

Can you prove it?

Remark: \( T_n(x) = n U_{n-1}(x) \)

Can you prove it?

\[\text{W} = 7.2 \quad \# 14, 5, 8, 12\]