1. Define $S = \{x \in \mathbb{R}^n : l \leq x \leq u\}$ where the inequalities are defined component-wise.
   
   (a) Writing $l = (l_1,\ldots,l_n)^T$ and $u = (u_1,\ldots,u_n)^T$, explain why solving
   $$\min_{x \in \mathbb{R}} |x - z|$$
   s.t. $l_i \leq x \leq u_i$
   for $i = 1,\ldots,n$ is sufficient to solve for
   $$\min_{x \in \mathbb{R}^n} ||x - z||$$
   s.t. $l \leq x \leq u$

   (b) Give a closed form expression for the projection of $z \in \mathbb{R}^n$ onto $S$. Carefully justify each step.

2. The statement of (a special case of) the Hilbert Projection Theorem is that for a nonempty closed convex subset $C$ of $\mathbb{R}^n$, the projection of any vector $x \in \mathbb{R}^n$ onto $C$ exists and is unique.
   
   (a) Show that existence does not need to hold if the subset $C$ is not closed.
   (b) Show that uniqueness does not need to hold if the subset $C$ is not convex.
   (c) Prove the uniqueness assertion of the Hilbert Projection Theorem on $\mathbb{R}^n$.

   Hint: You just showed that you need to use convexity.

3. Suppose that, as part of your minimization algorithm, your computer needs to solve the linear system
   $$\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
   
   (a) Show that when $\epsilon \neq 1$, the unique solution is given by
   $$x_1 = -x_2 = \frac{1}{\epsilon - 1}$$
   
   (b) Suppose your computer is precise to only 6 digits. Set $\epsilon = 10^{-6}$. Use Gaussian elimination and conclude that
   $$x_2 = \frac{-10^6}{1 - 10^6}$$
   
   What number does your computer solve $x_2$ as?
   
   (c) Your computer will use that value of $x_2$ and the first row of your linear system to now solve for $x_1$. What value does it solve $x_1$ as?
   (d) How does it compare to your answer from part a)?

4. Let $B > 0$, $f$ be a differentiable function on $\mathbb{R}^n$ and define
   $$q(x + p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T B p$$
   
   Prove that any minimizer of this function must be a descent direction for $f$ at $x$. 

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