Numerical Integration based on Interpolation

Idea: some functions are easy to integrate analytically, e.g.: 
- $f(x) = e^x$
- $f(x) = \sin x$

Others, like $f(x) = e^{x^2}$ or $g(x) = \int_0^x \sin(x^2) \, dx$ may not be.

Goal: Want a method to approximate 
$$\int_a^b f(x) \, dx$$ 
by using only $f(x_i), i=0, \ldots, n$.

Reasonable approach: use polynomial interp. 

to write $f(x) \approx p(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$

Then "hope" that $\int_a^b f(x) \, dx \approx \int_a^b p(x) \, dx$
So \( \int_{a}^{b} f(x) \, dx = \sum_{i=0}^{n} f(x_i) \ell_i(x) \, dx \)

\[ = \sum_{i=0}^{n} \int_{a}^{b} f(x_i) \ell_i(x) \, dx \]

exchange order \( a \to b \)

\[ = \sum_{i=0}^{n} f(x_i) \int_{a}^{b} \ell_i(x) \, dx \]

\( f(x_i) \) is a number

\[ = \sum_{i=0}^{n} A_i f(x_i) \]

In fact, you already know some examples of this

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**Trapezoid rule:** When \( n=1 \) & \( x_0=a, x_1=b \)

we have \( \ell_0(x) = \frac{x_1-x}{x_1-x_0} = \frac{b-x}{b-a} \)

\[ \ell_1(x) = \frac{x_0-x}{x_0-x_1} = \frac{x-a}{b-a} \]

\[ \Rightarrow A_0 = \int_{a}^{b} \frac{b-x}{b-a} \, dx = \frac{b(b-a)}{b-a} - \frac{(b-a)^2}{2(b-a)} = b - \frac{b+a}{2} \]

\[ A_0 = \frac{b-a}{2} \]

\[ A_1 = \frac{b-a}{2} \text{ (similarly)} \]

\[ \Rightarrow \int_{a}^{b} f(x) \, dx \approx \frac{b-a}{2} [f(a) + f(b)] \]
Its error term is \(-\frac{1}{12}(b-a)^3f''(\xi)\) for some \(\xi \in (a,b)\) (Proof uses MVT for integrals).

Composite trapezoidal rule:

1. Subdivide \([a, b]\) into \(n\) pieces using \(a = x_0 < x_1 < x_2 < \cdots < x_n < b\).

2. Use the trapezoidal rule for each piece

\[
\Rightarrow \int_a^b f(x) \, dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x) \, dx
\approx \frac{(x_i - x_{i-1})}{2} [f(x_i) + f(x_{i-1})]
\]

\[
\Rightarrow \int_a^b f(x) \, dx \approx \frac{1}{2} \sum_{i=1}^{n} (x_i - x_{i-1})(f(x_{i-1}) + f(x_i))
\]
Comp. Trap. rule with equal spacing

when all the subintervals of \([a, b]\) are the same length \(h = x_i - x_{i-1}\)

\[
\int_a^b f(x) \, dx \approx \frac{h}{2} \sum_{i=1}^{n} (f(x_{i-1}) + f(x_i))
\]

\[
= \frac{h}{2} \left[ f(a) + 2 \sum_{i=1}^{n-1} f(a + ih) + f(b) \right]
\]

Error term \(= -\frac{1}{12} (b-a) h^2 f''(\xi), \xi \in (a, b)\)

Back to the general, non-composite case

Recall:

\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^{n} A_i f(x_i)
\]

Since for polynomials of degree \(\leq n\)

\[
F(x) = \sum_{i=0}^{n} f(x_i) \ell_i(x)
\]

then our integration formula is exact for poly of deg \(\leq n\).
This observation allows us to find the \( A_i \)'s "easily" by the method of undetermined coefficients.

**Example:** \( N = 2 \), \([a, b] = [0, 1] \) & \( x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1 \)

\[
\Rightarrow \int_0^b f(x) \approx A_0 f(0) + A_1 f(1) + A_2 f(2)
\]

Formulas is exact for poly. of degree \( \leq 2 \)

So

\[
\int_0^1 f(x) \, dx = A_0 + A_1 + A_2
\]

\[
\int_0^{1/2} x \, dx = A_0 \cdot 0 + A_1 \cdot \frac{1}{2} + A_2
\]

\[
\int_0^{1/3} x^2 \, dx = A_0 \cdot 0 + A_1 \cdot \frac{1}{4} + A_2
\]

3 eq'n & 3 unknowns \( \Rightarrow \) \( A_1 = \frac{2}{3} \), \( A_2 = \frac{1}{6} = A_0 \)

\& \[
\int_0^b f(x) \, dx \approx \frac{1}{6} f(0) + \frac{2}{3} f(1) + \frac{1}{6} f(1)
\]
Simpson's Rule

Repeat the same calculation but with arbitrary \([a, b]\) and \(x_0 = a, x_1 = \frac{a + b}{2}, x_2 = b\) to get

\[
\int_a^b f(x)\,dx \approx \frac{b-a}{6} \left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)
\]

Remark: should be exact for poly. of degree \(\leq 2\) but is also exact for poly. of degree \(\leq 3\)

\[
\text{error term} = -\frac{1}{90} \left[ \frac{b-a}{2} \right]^5 f^{(4)}(\xi), \quad \xi \in (a, b)
\]

Composite Simpson's rule can also be used (see book)

Error Analysis

Want an expression for the error in numerical integration; that is, we want an expression for

\[
\int_a^b f(x) - \sum_{i=0}^{n} A_i f(x_i)
\]
Recall: \( A_i = \int_a^b e_i(x) \, dx \) where \( e_i(x) \) comes from the Lagrange interpolating poly:

\[
P(x) = \sum_{i=0}^{n} f(x_i) \ell_i(x)
\]

Also:

\[
f(x) - P(x) = \frac{1}{(n+1)!} \int_{x_0}^{x_{n+1}} f^{(n+1)}(y) \prod_{i=0}^{n} (x - x_i) \, dy
\]

So:

\[
\int_a^b f(x) \, dx - \sum_{i=0}^{n} A_i f(x_i) = \frac{1}{(n+1)!} \int_{a}^{b} f^{(n+1)}(x) \prod_{i=0}^{n} (x - x_i) \, dx
\]

can't do much about this as it depends on the \( f \) we're numerically integrating. Reasonable to just use an upper bound:

\[
M = \max_{x \in [a,b]} |f^{(n+1)}(x)|
\]

or \( M \geq \ldots
\]

Then:

\[
\left| \int_a^b f(x) \, dx - \sum_{i=0}^{n} A_i f(x_i) \right| \leq \frac{M}{(n+1)!} \int_{a}^{b} \prod_{i=0}^{n} |x - x_i| \, dx
\]