Initial Value Problems for ODE’s (IVP’s)

We want numerical methods for solving ODE’s of the form:

\[ \frac{dy}{dt} = f(t, y) \]

for \( a \leq t \leq b \) with \( y(t_0) = y_0 \)

i.e., we want numerical methods to approximate the solution \( y(t) \) of (\( \ast \)).

More generally:

\[ y^{(n)} = f(t, y', y'', ..., y^{(n-1)}) \]

for \( t \in [a, b] \) with \( y(a) = \alpha_1, y'(a) = \alpha_2, \ldots, y^{(n-1)}(a) = \alpha_n \)

and

System of ODEs

\[ \frac{dy_i}{dt} = f_i(t, y_1, y_2, ..., y_n); \quad i = 1, ..., n \]

with \( t \in [a, b] \), \( y_i(a) = \alpha_i; \quad i = 1, ..., n \)
One basic idea we will follow is to approximate the sol’n at certain pts \( \tilde{y}(t_1), \ldots, \tilde{y}(t_m) \) and use interpolation to get the function \( \tilde{y}(t) \). 

Before we dive in, we need some background theory.

**Def’n:** \( f(t,y) \) is Lipschitz in \( y \) on \( D \subset \mathbb{R}^2 \), with constant \( L \), if

\[
|f(t,y_1) - f(t,y_2)| \leq L |y_1 - y_2| \\
\forall (t,y_1), (t,y_2) \in D.
\]
Example 8 Let $f(t, y) = t/|y|$ for $D = [0,2] \times [-3,3]$ is Lipschitz because

$$|f(t, y_1) - f(t, y_2)| = |t| |y_1| - |y_2| |\leq 2 |y_1 - y_2|$$

Lipschitz const.

**Definition:** $D \subset \mathbb{R}^2$ is convex if $\forall (t, y), (t_2, y_2) \in D$ we have

$$(1-\lambda) (t, y_1) + \lambda (t_2, y_2) \in D$$

$\forall \lambda \in [0,1]$, not a convex set.
Theorem 1: If \( f : \mathbb{D} \subset \mathbb{R}^2 \to \mathbb{R} \) satisfied

\[
\frac{\partial f}{\partial y}(t, y) \leq \gamma \quad \forall (t, y) \in \mathbb{D}
\]

then \( f \) is Lipschitz on \( \mathbb{D} \) with constant \( \gamma \).

\[\uparrow \text{Sufficient cond. for a f to be Lipschitz}\]

IVP Existence & Uniqueness of Solns:

Theorem 1: (Existence)

If \( F \) is cont's on the rectangle

\[ R = \{ (t, y) : |t - t_0| \leq \alpha, \ |y - y_0| \leq \beta \} \]

then the IVP \((*)\) has a sol'n \( y(t) \) for

\[ |t - t_0| \leq \min (\alpha, \frac{\beta}{M}) \]

\[ M = \max_{(t, y) \in R} |F| \]
Example: Show that the IVP
\[
\begin{align*}
\dot{y} &= (t + \sin y)^2 \\
y(0) &= 3
\end{align*}
\]
has a soln on \( t \in [-1, 1] \).

Soln: The function \( f = (t + \sin y)^2 \) is cont'd on \( R = \{(t, y) : \quad t \leq 0 \}
\]
(\( y - 3 \leq \beta \))
(\( \text{This is true for all } a, \beta \gg 0 \))
and \( \max_{(t, y) \in R} |f(t, y)| \leq (a + 1)^2 \)
so the IVP has a soln for
\( |t| \leq \min(a, \frac{\beta}{(a + 1)^2}) \). Pick \( a = 1 \) & \( \beta = 4 \)
\( \Rightarrow \) IVP has a soln for \( |t| \leq 1 \).
(By Theorem 1)
Theorem 2 (Uniqueness):

If $f$ and $\frac{df}{dy}$ are cont's in $R = \{(t, y), -\alpha \leq (t-t_0) \leq \alpha, \beta \leq (y - y_0) \leq \beta\}$

then the IVP

$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

has a unique sol'n for

$$\mid t - t_0 \mid \leq \min (\alpha, \beta/M)$$

$M = \max |f| (y, t) \in R$

Note that the interval may be smaller than the base of the rectangle $R$.

Here is a different theorem:
Theorem: Let \( D = [a, b] \times \mathbb{R} \) and let \( f \) be continuous on \( D \). If \( f \) is Lipschitz in the variable \( y \) on then the IVP

\[
\begin{align*}
    y'(t) &= f(t, y), \quad t \in [a, b] \\
    y(a) &= \alpha 
\end{align*}
\]

has a unique sol'n \( y(t) \) for \( t \in [a, b] \)

Example: Use the theorem to show that \( \exists \) a unique solution to

\[
\begin{align*}
    y'(t) &= 1 + t \sin(ty), \quad 0 \leq t \leq 2 \\
    y(0) &= 0 
\end{align*}
\]

Sol'n: The function \( f(t, y) = 1 + t \sin(ty) \) is continuous on \( [0, 2] \times \mathbb{R} \) and has \( |\frac{\partial f}{\partial y}| = |t^2 \cos(ty)| \leq 4 \) on \( t \in [0, 2] \)

\( \Rightarrow \) \( f \) is Lipschitz with const 4

so by the theorem, the IVP has a unique sol'n.
Well Posedness:

The IVP \[ \frac{dy}{dt} = f(y, t), \quad t \in [a, b] \]
\[ y(a) = \lambda \]
is well-posed if

1. There exists a unique solution \( y(t) \)

2. The perturbed problem

\[ \left\{ \begin{array}{l}
\frac{dz}{dt} = f(z, t) + \delta(t), \quad t \in [a, b] \\
z(a) = \lambda + \delta_0
\end{array} \right. \]
also has a unique solution \( z(t) \) with

\[ |z(t) - y(t)| < k\epsilon \]

for any continuous \( \delta(t) \) with \( |\delta(t)| < \epsilon \) and \( \delta_0 < \epsilon \) for all \( t \in [a, b] \).
Why well-posedness?

- Modeling errors
- Round-off errors
- Measurement errors...

**Theorem:** If $f$ is cont. on $D$ where $D = [a, b] \times \mathbb{R}$, and if $f$ is Lipschitz on $y$ on $D$, the IVP:

$$\begin{cases} \frac{dy}{dt} = f(t, y), \quad t \in [a,b] \\ y(a) = x \end{cases}$$

is well posed.

**Euler's Method:**

$$\begin{cases} \frac{dy}{dt} = f(t, y), \quad t \in [a,b] \\ y(a) = x \end{cases}$$
Pick \( N+1 \) equispaced points \( t_0, \ldots, t_N \) with \( t_i = a + (i-1)h \), for \( h = \frac{b-a}{N} \)

and note that \( e^{c(t_i, t_i)} \)

\[
y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i)
\]

\[
w_{i+1}, \quad w_i, \quad h f(t_i, w_i) \text{ approx with } \frac{d}{dt}
\]

\[
\Rightarrow \begin{cases} 
  w_0 = x \\
  w_{i+1} = w_i + h f(t_i, w_i) 
\end{cases} \text{ for } i = 0, \ldots, N-1
\]

\[\text{Difference Eq'n.}\]

Now, we have approx. values of \( y \) at \( t_0, \ldots, t_N \), so we can interpolate to find an approx. \( y \).
Example: \[ \begin{array}{l}
\frac{dy}{dt} = y - t^2 + 1 \\
y(0) = 0.5
\end{array} \quad \text{for} \\ t \in [0,2] \]

Pick \( h = 1 \) & use Euler’s method:

\[ \begin{align*}
t_0 &= 0, & t_1 &= 1, & t_2 &= 2 \\
\omega_0 &= 0.5
\end{align*} \]

\[ \begin{align*}
\omega_1 &= \omega_0 + h f(t_0, \omega_0) \\
&= 0.5 + 1 \times (0.5 - 0^2 + 1) = 2
\end{align*} \]

\[ \begin{align*}
\omega_2 &= \omega_1 + h f(t_1, \omega_1) \\
&= 2 + 1 \times (2 - 1^2 + 1) = 4
\end{align*} \]

Error Bounds

Theorem: If \( f \) is Lipschitz continuous on \( D = [a,b] \times \mathbb{R} \) with const. \( L \) and if

\[ |y''(t)| \leq M \quad \forall t \in [a,b] \]

where \( y \) is the unique sol’n to

\[ \begin{array}{l}
\frac{dy}{dt} = f(t, y) \\
y(a) = y_0
\end{array} \]

then

\[ |y(t_i) - \omega_i| \leq \frac{h M}{2L} \left[ e^{Li} - 1 \right] \]
Higher order Taylor methods:

Local Truncation error:

In Euler’s method, we computed

\[ w_{i+1} = w_i + h f(t_i, w_i) \]

More generally, we may use an iteration of the form

\[ w_{i+1} = w_i + h \phi(t_i, w_i) \]

for some \( \phi \) we choose.

The local truncation error can be defined as

\[ \varepsilon_{i+1}(h) = w_{i+1} - (y_i + h \phi(t_i, w_i)) \]
Example 3: Euler's method has

\[ z_{i+1}(h) = y_{i+1} - y_i - hf(t_i, y_i) \]

\[ = \frac{h^2}{2} y''(\xi_i) \quad \xi_i \in [t_i, t_{i+1}] \]

\[ = O(h^3) \quad \text{as } \ h \to 0 \]

This suggests improving upon Euler's method by using higher order Taylor approximations:

\[ y(t_{i+1}) = y(t_i) + \sum_{j=1}^{n} \frac{h^j}{j!} y^{(j)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i) \]

\[ y'(t) = f(t, y(t)) \]

\[ y''(t) = f'(t, y(t)) \]

\[ y^{(k)}(t) = f^{(k-1)}(t, y(t)) \]
\[(*) \Rightarrow (**)
\]
\[y(t_{i+1}) = y(t_i) + \sum_{i=1}^{n} \frac{h^i}{i!} f^{(i-1)}(t_i, y(t_i))
\]
\[+ \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))
\]

To get our numerical method.

If we ignore the error term, we have

\[\text{Taylor methods of order } n \geq 1\]

\[
\begin{cases}
\text{ } & w_0 = \alpha \\
\text{ } & w_{i+1} = w_i + h \frac{f^{(n)}(t_i, w_i)}{n!}
\end{cases}
\]

\[f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)
\]

(Emuler = Taylor & order 1)
Example: Derive Taylor's method of order 2 with \( h=1 \) for the IVP

\[
\begin{cases}
    y' = y - t^2 + 1 & t \in [0, 2] \\
    y(0) = 0.5
\end{cases}
\]

Solution: we have \( f(t, y) = y - t^2 + 1 \) and we need to compute

\[
f'(t, y) = \frac{d}{dt} \left( y(t) - t^2 + 1 \right)
\]

always remember that \( y(t) \) is a R+ of \( t \!'

\[
= y'(t) - 2t
\]

So \( T^2(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) \)

\[
= w_i - t_i^2 + 1 + \frac{h}{2} \left( w_i^2 - t_i^2 - 2t_i + 1 \right)
\]

\[
= (1 + \frac{h}{2})(w_i - t_i^2 + 1) + -ht_i
\]
\[
\begin{align*}
\Delta \frac{w_0}{0.5} & \quad T(t_i, w_i) \\
\Delta w_{i+1} = w_i + h \left[ \left( 1 + \frac{h}{2} \right) (w_i - t_i^2 w_i) - ht_i \right]
\end{align*}
\]

**Theorem:** Taylor's method of order \( n \) has local truncation error \( O(h^n) \) provided the solution \( y \in C^n[a, b] \).

**Advantages:** high order local truncation error

**Disadvantage:** Must compute and evaluate derivatives of \( f(t, y) \).