BVP's: Shooting Methods for Linear Functions

Consider the linear two-point BVP:
\[ \begin{cases} x'' = w(t) + v(t)x + u(t)x' \\ x(a) = \alpha , \ x(b) = \beta \end{cases} \]  \hspace{1cm} (\ast)

Suppose that \( u(t), v(t), w(t) \) are all cont. on \([a, b]\).

Now suppose we solve (as we do in shooting methods) the BVP \( \ast \) by modifying it to an IVP to obtain the 2 solns: \( x_1(t) \) & \( x_2(t) \).

where \[ \begin{cases} x_1(a) = \alpha \\ x_1'(a) = z_1 \\ x_2(a) = \lambda \alpha \\ x_2'(a) = z_2 \end{cases} \]

Let \( y(t) = \lambda x_1(t) + (1-\lambda)x_2(t) \)

\( y \) solves \[ \begin{cases} y'' = f(t, y, y') \\ y(a) = \alpha , \ y'(a) = \lambda z_1 + (1-\lambda)z_2 \end{cases} \]
Now, we simply pick \( \lambda \) so that we solve the BVP, i.e., \( y(b) = \beta \)

\[
y(b) = \lambda x_1(b) + (-\lambda) x_2(b) = \beta
\]

\[
\Rightarrow \lambda = \frac{\beta - x_2(b)}{x_1(b) - x_2(b)}
\]

So, to solve linear BVP's like (*)

1. Solve the 2 IVP's (numerically)

\[
\begin{align*}
x''(t) &= f(t, x, x') \\
x(a) &= \alpha, \quad x'(a) = 0
\end{align*}
\]

\( \Rightarrow x_i(t_i) \quad i = 1, 2, \ldots \)

\[
\begin{align*}
x''(t) &= f(t, x, x') \\
x(a) &= \alpha, \quad x'(a) = 1
\end{align*}
\]

\( \Rightarrow x_2(t_i) \quad i = 1, 2, \ldots \)

2. Set \( \lambda = \frac{\beta - x_2(b)}{x_1(b) - x_2(b)} \)

3. The solution is approx. by

\[
y(t_i) = \lambda x_1(t_i) + (-\lambda) x_2(t_i)
\]
BVP's : Shooting / Newton’s Method

Want to solve

\[ \begin{align*}
&x'' = f(t, x, x') \\
&x(a) = \alpha, \quad x(b) = \beta
\end{align*} \]

Instead we solve

\[ \begin{align*}
&x'' = f(t, x, x') \\
&x(a) = \alpha, \quad x'(a) = z
\end{align*} \]  \hspace{1cm} \text{(1)}

with solution \( x_z(t) \) and error

\[ 
\phi(z) = x_z(b) - \beta
\]

L: non-linear equation in \( z \)

\[ \Rightarrow \text{Newton's method,} \]

\[ Z_{n+1} = Z_n - \frac{\phi(Z_n)}{\phi'(Z_n)} \]
Question

But we don't know $\phi(z)$ explicitly, how do we get $\phi'(z)$?

Answer

\[ \phi(z) = x_z(b) - \beta \Rightarrow \phi'(z) = \frac{dx_z(b)}{dz} \]

where

\[
\begin{cases}
  x'' = f(t, x, x') \\
  x_z(a) = a, \quad x_z'(a) = z
\end{cases} \tag{2}
\]

Great! But how do we get $\frac{dx_z(b)}{dz}$?

Answer

Differentiate (2) w.r.t. $z$!

\[
\begin{align*}
  \frac{d}{dz} x_z'' &= \frac{df}{dt} \frac{dx}{dz} + \frac{df}{dx} \frac{dx}{dz} + \frac{df}{dx'} \frac{dx'}{dz} \\
  \frac{d}{dz} x_z(a) &= 0 \quad , \quad \frac{d}{dz} x_z'(a) = 1
\end{align*}
\]
Rewriting this with \( v := \frac{dx^2}{dx} \)

\[
\Rightarrow v'' = \frac{\partial f}{\partial x_2} \cdot v + \frac{\partial f}{\partial x_1} \cdot v' - (3)
\]

This is an IVP, called the first variational eq’n

So now, you (numerically) solve (2) with initial cond \( x'(a) = z_n \)

and use this sol’n \( (x_{z_n}) \) to solve (3) \( \Rightarrow \) have \( \phi'(z_n) = v(b) \)

\( \Rightarrow \) use in Newton’s metl.

to get \( z_{n+1} \) and repeat
Multiple Shooting

Want to solve

\[\begin{align*}
    x'' &= f(t, x, x') \\
    x(a) &= \alpha, \quad x(b) = \beta
\end{align*}\]

\(\Rightarrow\) solve two IVP's

\[\begin{align*}
    x_1'' &= f(t, x_1, x_1') \quad a \leq t \leq c \\
    x_1(a) &= \alpha, \quad x_1'(a) = \beta_1
\end{align*}\]

\[\begin{align*}
    x_2'' &= f(t, x_2, x_2') \quad c \leq t \leq b \\
    x_2(b) &= \beta, \quad x_2'(b) = \beta_2
\end{align*}\]

For this one, we decrease \(t\)

Idea: Adjust \(\beta_1, \beta_2\) till the function

\[x(t) = \begin{cases} 
    x_1(t) & t \in [a, c] \\
    x_2(t) & t \in [c, b]
\end{cases}\]
solves the problem with

\[ x_1(c) = x_2(c) \, \& \, x'_1(c) = x'_2(c) \]

We can thus define the function

\[ \phi(z_1, z_2) = \begin{pmatrix} x_1(c) - x_2(c) \\ x'_1(c) - x'_2(c) \end{pmatrix} \]

both are functions of \( z_1, z_2 \)

Want the non-linear \( f^T \)

\[ \phi(z_1, z_2) = 0 \]

\[ \Rightarrow \text{Newton's method (in 2 variables)} \]

(17013)
Boundary Value Problems: Finite Differences

Idea: Discretize the $t$-axis
\[ t_i, i = 1, 2, \ldots, n \]
- Use approximations to the derivatives

Recall: \[ x'(t) \approx \frac{x(t+h) - x(t-h)}{2h} \]
\[ O(h^2) \text{ error} \]

\[ x''(t) \approx \frac{x(t+h) - 2x(t) + x(t-h)}{h^2} \]
\[ O(h^2) \text{ error} \]

So now instead of solving

\[ \begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \ x(b) = \beta \end{cases} \]

we solve (with $t_i = a + ih$, $i = 0, \ldots, n+1$)
\[ y_i = y(t_i) \]
Due to the function $f$, this is potentially a non-linear system of equations in $y = (y_0, \ldots, y_n)$ (we can use methods from 1703 to solve it).

**Finite Differences: The Linear Case**

When

$f(t, x, x') = u(t) + v(t)x + w(t)x'$

we have

\[
\begin{align*}
\frac{y_0}{h^2}(y_{i+1} - 2y_i + y_{i-1}) &= u_i + v_i y_i + \frac{w_i(y_i - y_{i-1})}{h} \\
\end{align*}
\]

(see next page)
linear in $y$

\begin{align*}
(1 - \frac{1}{2}h\omega_i)y_{i-1} + (2 + h^2\omega_i)y_i + (1 + \frac{1}{2}h\omega_i)y_{i+1} &= -\frac{h^2\omega_i}{a_i-1} \\
\frac{c_i}{d_i} &= \frac{b_i}{c_i}
\end{align*}

\Rightarrow \text{ we can write this in matrix form as } A\vec{y} = \vec{b} \Rightarrow

\begin{bmatrix}
  d_1 & c_1 & & & \\
  a_2 & d_2 & c_2 & & \\
  & & \ddots & \ddots & \\
  & & & a_{n-1} & d_{n-1} & c_{n-1} \\
  & & & & a_n & d_n
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_{n-1} \\
  y_n
\end{bmatrix}
= \begin{bmatrix}
  b_1 - a_0x_0 \\
  b_2 \\
  \vdots \\
  b_{n-1} - c_{n-1}x_{n-1}
\end{bmatrix}

\text{Tri-diagonal } \Rightarrow \text{ fast sol'n (170 A)}

\text{Theorem: If } v(t) > 0 \& w(t) \in C[0,a] \text{ then as } h \to 0, \text{ sol'n of } A\vec{y} = \vec{b} \text{ converges to sol'n of BVP}
Theorem on Existence & Uniqueness of solutions to BVP's:

\[ \begin{align*}
\frac{d^2x}{dt^2} &= f(t, x, x') \\
C_{11} x(a) + C_{12} x'(a) &= C_{13} \\
C_{21} x(b) + C_{22} x'(b) &= C_{23}
\end{align*} \]

has a unique solution on \([a, b]\) if

1. \( f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x'} \) are continuous on \( D = [a, b] \times \mathbb{R} \)
2. \( \frac{\partial f}{\partial x} > 0, \left| \frac{\partial f}{\partial x} \right| \leq M, \left| \frac{\partial f}{\partial x'} \right| \leq M \) on \( D \)

and

3. \( |C_{11}| + |C_{12}| > 0 \)
   \( |C_{21}| + |C_{22}| > 0 \)
   \( |C_{11}| + |C_{21}| > 0 \)
   \( C_{11}C_{12} \leq 0 \leq C_{21}C_{22} \)

Can use on most BVP's we've seen.
Topics covered in Midterm 2

* Everything till now.
* Focus on:

  * Multistep methods
    * Adams - Bashforth
    * Adams - Moulton
    * Explicit/implicit

  * Error Analysis
    * convergence/stability/consist.
    * Truncation error

  * Systems & Higher order ODE's
    * Taylor series methods
    * Other methods

  * BVP's
    * Existence/Uniqueness
      * Shooting Methods
        * Linear fits
        * Secant/Newton
      * Finite Differences