# Statement of Research Interests

## Introduction

Partial differential equations (PDEs) are at the heart of many physical systems, including solid mechanics, biochemical processes, and astrophysical phenomena. The solution of the vast majority of these equations becomes analytically intractable when used to model real world problems, so finding numerical approximations to their solutions is a critical task. Devising general frameworks to characterize various problems is an important aspect of deciding when the problems are well-posed and admit solutions, as well as of developing reliable computational methods to solve them. Much of my research has been based on studying various PDE frameworks and designing computer software for the solution of PDE discretizations. I am currently a postdoc in a small, interdisciplinary research group at the University of California, San Diego directed by Dr. Michael Holst. Members of the group are funded by numerous agencies including the NSF, NIH, DOE, and DOD. I have developed and now contribute a breadth of knowledge in both analytic mathematical techniques and computational methods, as well as expertise in elliptic and hyperbolic systems. In the remainder of this statement I will describe a number of past, present, and planned projects, all pertaining to the core theme of applied nonlinear PDE research.

## Relevant Early Work

My broad understanding of analytical and numerical PDE solution techniques has developed over the course of my educational career, beginning with my undergraduate honors project, furthered by my master's and doctoral research, and supported by summer internship experience.

In my final year of undergraduate study, I undertook a project with Dr. William Kocay where I studied the field equations of general relativity, and developed a small program to trace the path of light near a rotating black hole by numerically solving the geodesic equation in Kerr space-time. Although this project only involved solving a system of ordinary differential equations (ODEs), it both gave me my first look at numerical analysis and at an application area with many mathematical problems which continue to interest me.

From there, I moved onto the more complex field of PDE solution techniques. Large, sparse linear systems are the outcome of many numerical approaches to solving PDEs, and the ability to solve them quickly and efficiently is a topic of much research. My master's research with Dr. Paul Saylor dealt mainly with the design of parallel linear system solvers for such problems. In particular, I developed a parallel version of adaptive Chebyshev iteration [4] for the scalable solution of large, sparse systems. Ultimately the software is meant to assist in the solution of a flux-limited diffusion problem in the simulation of collapsing stars [5]. Later, I continued studying other linear system solution techniques, spending two summers at Lawrence Livermore National Labs, working with Dr. Ulrike Meier Yang on parallel algebraic multigrid methods [6] for matrix problems coming from PDE systems such as elasticity. I contributed code to the HYPRE software package [7, 8], adding a parallel interface for such block systems. This broad knowledge of computational linear system solution techniques, both in large software packages and small research codes, has been invaluable in much of my work that has come since.

## Semigroup Theory for Constrained Evolution Systems

Many problems in geometric PDE theory, including electromagnetism and relativity, can be stated as seeking a Banach-valued function $u(t)$ such that

$$\begin{aligned} \frac{d}{dt}u(t) &= A(u(t)) \\ C(u(t)) &= 0 \\ u(0) &= u_0 \end{aligned}$$

where $A$ and $C$ are general nonlinear operators. Here, the first equation represents an evolution system, the second captures constraints, and the final is an initial condition. When considering the unconstrained, linear case, semigroup theory can often be applied to problems in this form, yielding a solution $u(t) = T(t)u_0$ where $T(t)$ is a semigroup of bounded linear operators [9] satisfying

$$T(0) = I \quad \text{and} \quad T(s+t) = T(s)T(t).$$

When constraints are present, the theory can be used to show [10] propagation of constraints, requiring only that $C(u_0) = 0$. However, at the discrete level, this propagation is unrealistic without some modification.

In my thesis [1], I studied and developed solution approaches based on implicit time discretization and constrained least-squares techniques for the spatial problem. Various techniques based on an augmented Lagrangian [11, 12] were considered for this method and found to give satisfactory results with exact constraint satisfaction and error in evolution on the same order as a method which ignores the constraints. I wish to continue to develop the theory to include the nonlinear case, as well as study other methods of handling the constraints from a more geometric standpoint.

## Adaptive Finite Element Methods for Nonlinear PDEs

Finite elements can be used to discretize many PDEs written in weak form. Specifically, let $\Omega \subset \mathbb{R}^d$, $H$ be some Hilbert space of functions defined over $\Omega$ (with dual $H^*$), $a : H \times H \to \mathbb{R}$ a bilinear form, and $f \in H^*$. The abstract Galerkin problem for linear time-independant PDEs is stated as: find $u \in H$ such that

$$a(u, v) = f(v)$$

for all $v \in H$. The finite element approach uses a triangulation (or other partitioning), $\mathcal{T}$, of $\Omega$ to define a finite-dimensional subspace $H_h \subset H$, and instead seeks $u_h \in H_h$ such that

$$a(u_h, v_h) = f(v_h)$$

for all $v_h \in H_h$. This finite-dimensional problem is translated into a matrix problem through use of a basis for $H_h$ and, under typical assumptions on the continuous problem and the triangulation, the approximate solution $u_h$ can be found efficiently and used to approximate the exact solution $u$.

In order to improve this approximate solution, the triangulation must be refined, enlarging the discrete space $H_h$. For the algorithm to be efficient, triangles where the error in the approximation is large should be marked for refinement. However, the error is an unknown quantity, and thus, it too must be estimated. The use of *a posteriori* error estimation [13] has become a popular approach, and the standard iteration for adaptive finite element methods is viewed in four stages,

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}.$$

Recent work [14, 15] has shown shown, under various assumptions on the different algorithmic stages, that this iteration convergences for linear PDEs.

I am currently working with Dr. Michael Holst and Dr. Jeffrey Ovall [3] on a specific error estimator for problems in three dimensions which is provably efficient and reliable up to an oscillation term. I am implementing the estimator in FETK [16] to numerically study the efficiency of this estimator and test its robustness for general linear and nonlinear elliptic PDEs. In the future, I plan to consider the general framework above for nonlinear systems of PDEs. In particular, we are considering the problem for the constraint equations of general relativity, as well as other nonlinear geometric elliptic PDEs. Use of specific finite spaces and development of error estimators may be needed to guarantee convergence for these problems.

# Space-Time Finite Element Methods

Time-dependant problems are classically discretized with space and time treated independently, leading to the need to solve a sequence of spatial problems (if time is discretized first) or a large system of ODEs (if space is discretized first). This approach may seem natural for some problems, but often, when adaptivity in space is necessary (when capturing a shock, for example), it can place unreasonable global requirements on the size of the time step. There has been significant work on local time-stepping techniques, such as the Runge-Kutta discontinuous Galerkin method [17], and these have been shown to work for a large class of problems. However, approaches where space and time are discretized simultaneously may have additional benefits.

In order for a space-time finite element approach to be stable and efficient, many requirements must be imposed on the mesh and finite element space. For efficiency, it should progress a moving front through time, allowing for a local solution at nodes which are progressed forward. The use of discontinuous Galerkin methods allows for such local solutions in a very natural way. However, the mesh must be such that it satisfies local time-stepping requirements to maintain causality and stability. Recent work on a method called tent pitching [18] shows promise of being an efficient and robust algorithm for generating space-time meshes which satisfy strict causality and allow for local refinement, coarsening, or mesh smoothing in a natural way.

I recently performed some simple testing in the one-dimensional setting. I would like to continue to study this method for nonlinear problems with discontinuous wave speeds (perhaps due to shocks) where it seems the algorithm may break down. Also, in order to fully resolve wave fronts through adaptivity, I wish to further develop local error estimates for space-time discontinuous Galerkin finite elements.

## Topologically Massive Gravity

Topologically massive gravity [19] is a 2+1 gravitational theory (2 spatial and 1 temporal dimension) where an extra Chern-Simons term has been added to the standard Lagrangian of general relativity. The theory has broad uses in quantum gravity as a model problem for high dimensional gravity and supergravity models. The field equations of TMG are easily stated as

$$G_{\mu\nu} + C_{\mu\nu} = 0,$$

where $G_{\mu\nu}$ is the Einstein tensor and $C_{\mu\nu}$ is the Cotton tensor. When this equation is written in terms of the metric tensor $g_{\mu\nu}$ it becomes a third-order nonlinear constrained PDE.

Along with Dr. Michael Holst and Dr. Gantumur Tsogtgerel [2], I have been studying the Cauchy problem for this system from a mathematical perspective. We have shown that a local-in-time solution exists under a particular (physically meaningful) gauge choice. There is still much work which can be done with these field equations from both analytical and numerical standpoints. I would like to study various finite element-based approaches to discretizing the problem, including how to handle the constraints, and how to deal with adaptivity in both space and time.

## Conclusions

My broad experience with nonlinear PDEs at both the analytical and numerical levels, coupled with my strong background in computation, gives me the ability to study and grasp many diverse problems in applied mathematics. I plan to continue to develop my understanding of nonlinear PDEs and finite element discretizations of them. Both theoretic and software-based approaches will continue to play a significant role in my research program.

## References

[1] R. Szypowski. *Least-Squares Finite Elements and Constrained Evolution Systems.* PhD thesis, UCSD, 2008.

[2] M. Holst, R. Szypowski, and G. Tsogtgerel. The Cauchy problem of cosmological topologically massive gravity in 2+1 dimensions, 2009. (preprint).

[3] M. Holst, J. Ovall, and R. Szypowski. An efficient, reliable and robust error estimator for elliptic problems in $\mathbb{R}^3$, 2009. (preprint).

[4] R.S. Varga. *Matrix iterative analysis.* Prentice-Hall, 1962.

[5] F.D. Swesty, D.C. Smolarski, and P.E. Saylor. A comparison of algorithms for the efficient solution of the linear systems arising from multigroup flux-limited diffusion problems. *The Astrophysical Journal Supplement Series*, 153:369–387, 2004.

[6] A. Brandt, S.F. McCormick, and J. Ruge. Algebraic multigrid (AMG) for sparse matrix equations. *Sparsity and its Applications*, pages 257–284, 1984.

[7] R.D. Falgout and U.M. Yang. hypre: A library of high performance preconditioners. In *Conference: International Conference on Computational Science (ICCS) 2002, Amsterdam (NL), 04/21/2002–04/24/2002*, 2001.

[8] V.E. Henson and U.M. Yang. BoomerAMG: A parallel algebraic multigrid solver and preconditioner. *Applied Numerical Mathematics*, 41(1):155–177, 2002.

[9] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer, 1983.

[10] H.R. Beyer. *Beyond Partial Differential Equations: On Linear and Quasi-Linear Abstract Hyperbolic Evolution Equations.* Lecture Notes in Mathematics. Springer, 2007.

[11] M.J.D. Powell. A method for nonlinear constraints in minimization problems. *Optimization*, pages 283–298, 1969.

[12] M.R. Hestenes. Multiplier and gradient methods. *Journal of Optimization Theory and Applications*, 4(5):303–320, 1969.

[13] I. Babuska and W.C. Rheinboldt. A-posteriori error estimates for the finite element method. *International Journal for Numerical Methods in Engineering*, 12(10):1597–1615, 1978.

[14] K. Mekchay and R.H. Nochetto. Convergence of adaptive finite element methods for general second order linear elliptic pdes. *SIAM Journal on Numerical Analysis*, 43(5):1803–1827, 2006.

[15] J.M. Cascon, C. Kreuzer, R.H. Nochetto, and K.G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method, 2007. (preprint).

[16] M. Holst. FETK. http://www.fetk.org/.

[17] B. Cockburn and C.W. Shu. The Runge-Kutta local projection P1-discontinuous Galerkin finite element method for scalar conservation laws. *IMA Preprint Series*, (388), 1988.

[18] A. Ungor and A. Sheffer. Tent-Pitcher: A meshing algorithm for space-time discontinuous Galerkin methods. In *Proc. 9th Int. Meshing Roundtable*, pages 111–122, 2000.

[19] S. Deser, R. Jackiw, and S. Templeton. Three-dimensional massive gauge theories. *Physical Review Letters*, 48(15):975–978, 1982.