SET® and AG(4,3)

Food for Thought

Robert Won

(Lafayette REU 2010 - Joint with M. Follett, K. Kalail, E. McMahon, C. Pelland)

“Partitions of AG(4,3) into maximal caps”, *Discrete Mathematics* (2014)
The card game SET®

- SET® is played with 81 cards. Each card is characterized by 4 attributes:
  - **Number**: 1, 2 or 3 symbols.
  - **Color**: Red, purple or green.
  - **Shading**: Empty, striped or solid.
  - **Shape**: Ovals, diamonds or squiggles.

- A set is three cards where each attribute is independently either all the same or all different.
The card game SET®

The number of attributes that are the same can vary.

Shape and shading are the same, color and number are different.

All attributes are different.
The card game SET®

- To start the game, twelve SET® cards are dealt face up.
- If a player finds a set, he takes it and three new cards are dealt.
- If there are no sets, three more cards are dealt. The three cards are not replaced on the next set, reducing the number back to twelve.
- The player who finds the most sets is the winner.

Image adapted from Davis and Maclagan “The Card Game SET”
The card game SET®

Can you find a set?
For affine geometry on a plane, there are three axioms:

1. There exist (at least) 3 non-collinear points.
2. Any two points determine a unique line.
3. Given a line \( \ell \) and a point \( P \) not on \( \ell \), there is a unique line through \( P \) parallel to \( \ell \).

The order of a finite geometry is the number of points on each line.

Using these axioms, we can draw \( AG(2, 3) \), the affine plane of order 3.
Finite affine geometry
Seeing geometry in SET®

• A deck of SET® cards is a finite affine geometry. The cards are the points; three points are on a line if those three cards form a set.

• This works because any two cards uniquely determine a third card that completes the set.
Seeing geometry in SET®

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Coordinatizing SET\textsuperscript{®}

- We can also think of a deck of SET\textsuperscript{®} cards as the vector space $\mathbb{F}_3^4$.
- Each attribute corresponds to a coordinate, which can take on one of three possible values

<table>
<thead>
<tr>
<th>Number</th>
<th>Color</th>
<th>Shading</th>
<th>Shape</th>
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</thead>
<tbody>
<tr>
<td>1→1</td>
<td>red→1</td>
<td>empty→1</td>
<td>oval→1</td>
</tr>
<tr>
<td>2→2</td>
<td>green→2</td>
<td>striped→2</td>
<td>diamond→2</td>
</tr>
<tr>
<td>3→0</td>
<td>purple→0</td>
<td>solid→0</td>
<td>squiggle→0</td>
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</tbody>
</table>
Coordinatizing SET®

• With this choice of coordinates:

- The first set has coordinates \((1, 0, 2, 1), (2, 1, 2, 1)\) and \((0, 2, 2, 1)\).
- The second set has coordinates \((2, 1, 0, 0), (1, 2, 1, 1), (0, 0, 2, 2)\).
- Three cards form a set if and only if
  • the vector sum is \(\vec{0}\) mod 3
  • they are of the form \(\vec{x}, \vec{x} + \vec{a}, \vec{x} + 2\vec{a}\) for some \(\vec{a} \neq \vec{0}\)
Parallel sets

• We can also see parallel lines as parallel sets.
• If the original set has any attribute all the same, the parallel set will also have the same attribute all the same.
• If any attributes are different in the set, you can lay the cards of the parallel set down so that each of those attributes cycle in the same way as in the original.
Finite affine planes of SET® Cards

[Images of SET® cards]

Connecting SET® to geometry
Finite affine planes of SET® Cards
Finite affine planes of SET® Cards

[Diagram of SET® cards]

Connecting SET® to geometry
Finite affine planes of SET® Cards
Finite affine planes of SET® Cards

This picture is sometimes called a magic square. Find all the sets in it!
Finite affine planes of SET® Cards

This picture is sometimes called a magic square. Find all the sets in it!
A finite affine hyperplane

Select any remaining card and construct two more magic squares. This creates a hyperplane.
A finite affine hyperplane

A line in $AG(3, 3)$:

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\end{array} \quad \begin{array}{ccc}
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The entire deck ($AG(4, 3)$)
Some easy counting

• How many cards (points) are there?
Some easy counting

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  \[81 = 3^4\]
Some easy counting

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• How many sets (lines) are there?
Some easy counting

• How many **cards** (points) are there?
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• How many **sets** (lines) are there?
  \[ 1080 = (81 \times 80)/3! = \binom{81}{2}/3 \]
Some easy counting

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  \[ 1080 = \frac{81 \times 80}{3!} = \frac{\binom{81}{2}}{3} \]

• How many sets through a given card are there?
Some easy counting

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  \[ 81 = 3^4 \]

- How many sets (lines) are there?
  \[ 1080 = \frac{81 \times 80}{3!} = \binom{81}{2}/3 \]

- How many sets through a given card are there?
  \[ 40 = \frac{80}{2} \]
Back to $\mathbb{F}_3^n$

- In $\mathbb{F}_3^n$, we define a line (a.k.a. an algebraic line) to be:
  - three points that sum to $\vec{0}$ mod 3
  - three points of the form $\vec{x}, \vec{x} + \vec{a}, \vec{x} + 2\vec{a}$ for some $\vec{a} \neq \vec{0}$
- Now we have linear (well, affine) algebra! The maps $\mathbb{F}_3^n \to \mathbb{F}_3^n$ taking lines to lines are precisely the affine transformations

$$\vec{x} \mapsto A\vec{x} + \vec{b}$$

for $A \in \text{GL}(n, 3)$
Complete caps

• A \textit{k-cap} is a collection of \( k \) points with no three collinear.
• A \textit{complete cap} is a cap for which any other point in the space makes a line with a subset of points from the complete cap.
• A \textit{maximal cap} is a cap of maximum size.
• In \( \mathbb{F}_3^2 \), maximal caps contain four points.

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\end{array}
\]
Complete caps

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- In \( \mathbb{F}_3^2 \), maximal caps contain four points.
Two complete caps in $\mathbb{F}_3^3$

A complete 8-cap:

A complete (and maximal) 9-cap:
Complete caps

• Two caps $c_1$, $c_2$ are called equivalent if there exists an affine transformation mapping $c_1$ to $c_2$.

• Fact: All maximal caps in $\mathbb{F}_3^n$ are equivalent for $n \leq 6$.
  
  $n = 4$, the Pellegrino cap (Hill – 1983)
  $n = 5$, the Hill cap (Edel, Ferret, Landjev, Storme – 2002)
  $n = 6$, (Potechin – 2008)

• Open question: $n > 6$?
An integer sequence

Denote by $M(n, 3)$ the size of a maximal cap in $\mathbb{F}_3^n$
An integer sequence

- Terry Tao’s blog: “Open question: best bounds for cap sets”
  (http://terrytao.wordpress.com/2007/02/23/open-question-best-bounds-for-cap-sets/)
- “Perhaps my favourite open question is the problem on the maximal size of a cap set — a subset of $\mathbb{F}_3^n$ which contains no lines...”

<table>
<thead>
<tr>
<th>$n =$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>$M(n, 3) \geq$</td>
<td>2</td>
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<td>9</td>
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<td>45</td>
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<td>236</td>
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<td>1008</td>
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<td>$M(n, 3) \leq$</td>
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<td>45</td>
<td>112</td>
<td>292</td>
<td>773</td>
<td>2075</td>
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- The best asymptotic bounds

$$(2.2174...)^n \leq M(n, 3) \ll 3^n / n$$
Maximal caps in $AG(4, 3)$

Remember this cap? Consider this card.
Maximal caps in $AG(4, 3)$

Theorem (F, K, M, P, -, 2014)  

(First observed by Gary Gordon) Every 20-cap in $AG(4, 3)$ consists of ten lines intersecting at one point with the point of intersection removed. We call this point the anchor point.
Maximal caps in $AG(4, 3)$

**Theorem (F, K, M, P, -, 2014)**

*Any two maximal caps in $AG(4, 3)$ with different anchor points intersect*
Partitioning $AG(4, 3)$

(Tony Forbes) $AG(4, 3)$ can be partitioned into 4 disjoint 20-caps and their anchor point.
Partitioning $AG(4, 3)$
Partitioning $AG(4, 3)$

Are all partitions of $AG(4, 3)$ equivalent?
Linear transformations

When the anchor point is fixed at $\vec{0}$, affine transformations are linear transformations. Here’s one:
Spot the difference

Consider these two caps with respect to our favorite cap, $S$.

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Spot the difference

- The first is “1-completable”.
- The second is “2-completable”.

![Diagram showing two sets of partitions with different completions.](image-url)
Spot the difference

6-completables, too!
Completability

• 198 caps disjoint from $S$.

• With respect to our favorite cap, $S$:
  • 36 1-completable caps
  • 90 2-completable caps
  • 72 6-completable caps

• Every partition consists of:
  • $S$, 1-completable, 6-completable, 6-completable
  • $S$, 2-completable, 6-completable, 6-completable
Linear transformations

Theorem (F, M, K, P, -, 2014)

Let $T$ be an affine transformation fixing $S$:

$$T(n\text{-comp}) \text{ is an } n\text{-comp, } n \in \{1, 2, 6\}$$

No affine transformations exist between 1-completables and 2-completables.
Partition classes

- 216 different partitions of $AG(4, 3)$ with $S$.
  \[36 \times 1 + 90 \times 2 = 216\]

- Two equivalence classes (no affine transformations):
  - $E_1$: 36 partitions \{S, 1-comp, 6-comp, 6-comp\}
  - $E_2$: 180 partitions \{S, 2-comp, 6-comp, 6-comp\}

- Each 6-completable once in $E_1$ and five times in $E_2$. 
Linear transformations of $E_2$

Suppose $D_2 \in E_2$, and let $S_2$ be the 2-completable of $D_2$.

- 8 linear transformations fix $D_2$ cap-wise ($\cong \mathbb{Z}_4 \times \mathbb{Z}_2$).
- 8 linear transformations fix $S$ and $S_2$ and switch 6-complettables.
- Thus, a group of order 16 fixes $D_2$ set-wise ($\cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4$).

- Another set of 16 linear transformations fix $S$ and $S_2$ but send 6-complettables to two new 6-complettables.
- Thus, group of order 32 fixing $S$ and $S_2$ ($\cong (\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$).
Linear transformations of $E_1$

Suppose $D_1 \in E_1$, and let $S_1$ be the 1-completable of $D_1$.

- 40 transformations fix $D_1$ cap-wise ($\cong \mathbb{Z}_4 \times D_5$).
- Also, 40 transformations fix $S$ and $S_1$ and switch 6-completables.

- Thus, group of order 80 fixing $S$ and $S_1$.
- Isomorphic to $\mathbb{Z}_{20} \rtimes \mathbb{Z}_4$. 

UC San Diego
Summary

• Every maximal cap in $AG(4, 3)$ consists of ten lines intersecting at an anchor point.
• $AG(4, 3)$ can be partitioned into four disjoint maximal caps and their anchor point.
• There are two equivalence classes of partitions.

Also interesting

• Building complete caps (Jordan Awan):
  http://webbox.lafayette.edu/~mcmahone/capbuilder.html