Math 3C Summer II 2015 - Final Exam

Instructions: Put your name and PID on your blue book. Turn off and put away your cell phone. No calculators. You may use both sides of one 8.5 × 11 sheet of notes, but no other resources. Make sure your solutions are clear and legible, and show all your work. Credit may not be given for unreadable or unsupported answers. Write your solutions in your blue book, keeping the questions in order, and clearly indicating which problem is being solved. This exam is out of 75 points, with 4 possible extra credit points.

Question 0. (1 point) Carefully read all instructions above and make sure you have followed them all, as well as any other instructions given by your instructor during the exam.

Question 1.

(a) (4 points) Suppose \( \pi \leq \theta < 2\pi \) and \( \tan \theta = 2 \). Find \( \sin \theta \).

**Solution.** We draw the appropriate triangle. Notice that \( \pi \leq \theta < 2\pi \) so \( \sin \theta \) is negative. Hence, we conclude \( \sin \theta = -\frac{2}{\sqrt{5}} \).

(b) (4 points) Compute \( \sin (\tan^{-1} 2) \).

**Solution.** In this case, since \( \tan^{-1} 2 \) is an angle in the first quadrant, \( \sin \theta = \frac{2}{\sqrt{5}} \).

Question 2. Let \( f(x) = 3 \tan^{-1}(x) - 3 \).

(a) (2 points) What is the largest possible domain for \( f(x) \)?

**Solution.** \( f(x) \) has the same domain as \( \tan^{-1}(x) \): all real numbers \(( -\infty, \infty )\).

(b) (4 points) What is the range of \( f(x) \)?

**Solution.** Since \( \tan^{-1}(x) \) has range \(( -\frac{\pi}{2}, \frac{\pi}{2} ) \), therefore \( f(x) \) has range \(( -\frac{3\pi}{2} - 3, \frac{3\pi}{2} - 3 ) \).

(c) (4 points) Find all roots of \( f(x) \).

**Solution.** Roots of \( f(x) \) occur when \( \tan^{-1}(x) = 1 \), so \( x = \tan(1) \) is the only root.

Question 3. (4 points) Give the equation of the line that forms an angle of 30° with the x-axis and intersects the x-axis at \(( 3, 0 )\).

**Solution.** The line has slope \( \tan(30^\circ) \) and goes through \(( 3, 0 )\) so has form

\[
y = \tan(30^\circ)(x - 3).
\]
Question 4. (6 points) Let \( h(x) = 3 \cdot 5^{2x} \). Recall that we can also write \( h(x) \) in the form \( h(x) = c \cdot B^x \) or \( h(x) = c \cdot 2^{kx} \). What are \( c \), \( B \), and \( k \)?

**Solution.** Rewriting in the two forms, \( h(x) = 3 \cdot 25^x \) or \( h(x) = 3 \cdot 2^{\log_2(25)x} \). Hence \( c = 3 \), \( B = 25 \), and \( k = \log_2(25) \).

Question 5. Compute the following

(a) (2 points) \( \sec(-\frac{11\pi}{6}) = \frac{2}{\sqrt{3}} \)
(b) (2 points) \( \tan\left(\frac{15\pi}{4}\right) = -1 \)
(c) (2 points) \( \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \)
(d) (2 points) \( \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3} \)

Question 6. (4 points) Find a number \( a \) such that the following system has no solutions.

\[
\begin{align*}
2x + y &= 3 \\
ax - 2y &= 4
\end{align*}
\]

**Solution.** For a system of two linear equations in two variables to have no solutions, they must be non-intersecting parallel lines. Hence, we must choose \( a \) such that the graphs of the two lines are parallel, hence \( a = -4 \).

Question 7. The height \( h(t) \) of a thrown ball (in ft) at the time \( t \) seconds is given by:

\[
h(t) = -16t^2 + v_0t + h_0
\]

where \( v_0 \) is the initial velocity in ft/sec and \( h_0 \) is the initial height in ft. Suppose you throw a ball from a height of 4 ft with an initial velocity of 8 ft/sec.

(a) (2 points) At what time does the ball reach its maximum height?

**Solution.** We should complete the square to find the vertex of the parabola, which will give the maximum height and the time it is reached. Completing the square, \( h(t) = -16(t - \frac{1}{4})^2 + 5 \). The vertex of this parabola occurs at \( (\frac{1}{4}, 5) \), so the ball reaches its maximum height at time \( t = \frac{1}{4} \) seconds.

(b) (2 points) What height does the ball reach at its maximum?

**Solution.** By the above computation, the maximum height is 5 feet.

(c) (2 points) At what time does the ball hit the ground?

**Solution.** The ball hits the ground when \( h(t) = 0 \). Hence, we need to solve the equation \( 16t^2 - 8t - 4 = 0 \), which we can solve using the quadratic formula (and take only the positive solution) so the ball hits the ground at time

\[
t = \frac{8 + \sqrt{64 + 256}}{32} \text{ seconds}
\]
Question 8. (4 points) Solve for $x$:

$$\ln(x + 5) + \ln(x - 1) = 2.$$  

**Solution.** Combining the natural logs,

$$\ln((x + 5)(x - 1)) = 2.$$  

Now we see

$$x^2 + 4x - 5 = e^2$$

or

$$x^2 + 4x - (5 + e^2) = 0$$

which we can solve using the quadratic formula so

$$x = \frac{-4 \pm \sqrt{16 + 4(5 + e^2)}}{2}.$$  

One of these solutions is negative, so is not in the domain of $\ln(x - 1)$. Hence, the only solution is given by

$$x = \frac{-4 + \sqrt{16 + 4(5 + e^2)}}{2}.$$  

Question 9. (4 points) Suppose $g(x)$ has the form $g(x) = a \cos(bx) + c$. If $g(x)$ has period $\frac{3\pi}{4}$ and range $[-5, 1]$, find $a$, $b$, and $c$.

**Solution.** The amplitude is $\frac{1-(-5)}{2} = 3$ so $a = 3$. The period is $\frac{3\pi}{4} = \frac{2\pi}{b}$ so $b = \frac{8}{3}$. Finally, we see that the range has been shifted down by 2 so $c = -2$.

Question 10. (4 points) Suppose an 8 ft long ladder is propped against a wall. If the ladder makes an angle of $74^\circ$ with the ground, how high up the wall does the ladder reach?

**Solution.** Draw the appropriate triangle to see that the ladder reaches up the wall $8 \sin(74^\circ)$ feet.

Question 11. Determine whether the following functions are odd, even, or neither.

(a) (4 points) $f(x) = \cos(x + \pi)$

**Solution.** This function is equal to $-\cos(x)$ so is even.

(b) (4 points) $g(x) = \sin(3\pi/2 - x)$

**Solution.** This is equal to $\sin(\pi + \pi/2 - x) = -\sin(\pi/2 - x) = \cos(x)$ so is even.
**Question 12.** (4 points) Find all horizontal and vertical asymptotes of the rational function

\[ r(x) = \frac{3x^2 + 2x - 1}{16 - 4x^2}. \]

**Solution.** There are vertical asymptotes where the denominator is equal to 0, so \( x = \pm 4 \) are the two vertical asymptotes. The horizontal asymptotes correspond to behavior near infinity, so \( y = -\frac{3}{4} \) is a horizontal asymptote.

**Extra credit.** (4 points all-or-nothing) Let \( p(x) = x^3 - 3x^2 - 2x + 4 \). Find all roots of \( p(x) \). (**Hint:** Can you guess a root?)

**Solution.** We can check that \( p(1) = 0 \), so that 1 is a root of \( p(x) \). Hence, we can factor out \( x - 1 \) using polynomial division to see

\[ p(x) = (x - 1)(x^2 - 2x - 4). \]

We can then use the quadratic formula to find the remaining roots. The roots of \( p(x) \) are therefore

\[ 1, \frac{2 + \sqrt{4 + 16}}{2}, \frac{2 - \sqrt{4 + 16}}{2}. \]

Good work this quarter! If you are taking calculus next quarter, best of luck!