**Magnetic Graphs & Lifts**

A combinatorial graph $G = (V(G), E(G))$ is called simple if its vertex set is finite and its edge set contains no loops or multiple edges. A graph is called connected if there is at least one path connecting any two vertices. Throughout, we consider simple, connected graphs. If two vertices $u, v \in V$ are adjacent, we write $u \sim v$.

**Signatures**

The oriented edge set of a graph $G$ is given by $E^o(G) = \{(u, v), (v, u) : u, v \in V(G), u \sim v\}$.

A signature on a graph is a map $\sigma : E^o(G) \to \mathbb{Z}^*: \{u, v\} \to \sigma_{uv}$ satisfying the property $\sigma_{uv} = -\sigma_{vu}$. A pair $(G, \sigma)$ is called a magnetic graph.

![Figure: Various lifts from the preceding magnetic graphs.](image)

**What is optimal transport on graphs?**

Let $G = (V(G), E(G))$ be a simple connected graph equipped with the shortest-path metric $d$. Suppose one has two mass (probability) distributions defined on the vertices of a graph, say $\nu, \mu : V(G) \to \mathbb{R}$, then we may consider the question of how one can transport the mass $\mu$ to the mass $\nu$. This is formalized with the notion of a transport plan $\gamma$, a non-negative function which quantifies the amount of mass moved from vertex $u$ to vertex $v$. $\gamma(u, v)$ is the set of all admissible $\mu$, $\nu$-transport plans $\gamma$. Then the transport cost of $\mu$ and $\nu$ with respect to the metric $d$ (for the Wasserstein metric) may be formulated:

$$W_d(\nu, \mu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \sum_{uv \in E(G)} d(u, v)\gamma(u, v).$$

(1)

**Optimal transport on graphs** is the study of this quantity, others like it, and the transport plans which attain them.

Let $\nu \in V(G)$ be a fixed 'base vertex.' We define the Lipschitz space and its norm:

$$Lip^\sigma(G) = \{f : V(G) \to \mathbb{R} \mid f(u) = 0, \|f\|_{Lip} = \max_{uv \in E(G)} |f(u) - f(v)|\}$$

for each $f \in Lip^\sigma(G)$. If $f \in Lip^\sigma(G)$ and if $\|f\|_{Lip} \leq 1$, then $f$ is called an extreme point of the unit ball in $Lip^\sigma(G)$ (denoted $B_{Lip}^\sigma$). If $f$ is an extreme point of $B_{Lip}^\sigma$, then $g = 0$. If $(u, v) \in E(G)$, we say that $(u, v)$ is a satisfied by $f$ provided $\|f(u) - f(v)\| = 1$.

**Convex extreme points.**

Let $(G, \sigma)$ be an unbalanced graph, and $f \in B_{Lip}^\sigma$. Then $f$ is an extreme point of $B_{Lip}^\sigma$ if and only if the magnetic graph $H_f$ defined by the vertex set $V(G)$, the edge set $E(H_f) = \{(u, v) \in E(G) \mid (u, v) \text{ is satisfied by } f\}$, and which we equip with the same signature structure $\sigma$ as on $G$, is unbalanced on each of its connected components. Similarly, we may define a magnetic atom for every pair of adjacent vertices $u, v$, and the $\sigma$-Arens-Eells space to be $A^E: \sigma(G) := \text{span}\{m_{uv} \mid u, v \in V(G)\}$ equipped with the norm $\|m\|_E := \inf \left\{ \sum_i |a_i| \mid m = \sum_i a_i m_{u_i, v_i} \right\}$.

**Kantorovich duality.**

For an unbalanced, simple magnetic graph $(G, \sigma)$, the spaces $A^E(X)$ and $\text{Lip}^\sigma(X)$ are isometrically isomorphic.

**Open Questions**

(1) How can we further describe $\|\cdot\|_E$ in terms of the norm $\|\cdot\|_{\text{Lip}^\sigma}$ using the compression mapping?

(2) How can magnetic transport be interpreted as a physical process?

**Results**

In the case of a simple magnetic graph $(G, \sigma)$, we may consider two new normed spaces. The $\sigma$-Lipschitz space $\text{Lip}^\sigma(G)$ and its norm are defined by $Lip^\sigma(G) = \{f : V(G) \to \mathbb{R} \mid \|f\|_{Lip, \sigma} = \max_{uv \in E(G)} |f(u) - f(v)|\}$.

In $\text{Lip}^\sigma(G)$, if $f \in \text{Lip}^\sigma(G)$ with $\|f\|_{Lip, \sigma} \leq 1$, then $f$ is called an extreme point of the unit ball in $\text{Lip}^\sigma(G)$ (denoted $B_{Lip}^\sigma$) provided that for any $g \in \text{Lip}^\sigma(G)$, if it holds that $\|f + tg\| < 1$ for $t \in [-1, 1]$.

**Compression Transformation**

We define the linear compression mapping $C : A^E(G) \to A^E(G)$ by setting, for each $m \in A^E(G), u \in V(G)$,

$$C(m)(u) = \sum_{v \in V} \xi_{uv} m(v).$$

$C$ is in fact a surjective contraction onto the space $A^E(G)$, which we have the equation $\|m\|_E \leq \|C(m)\|_E$ for each $m \in A^E(G)$.

**Notes and References**