THE WEIGHTED SPECTRUM OF THE UNIVERSAL COVER AND AN ALON-BOPPANA RESULT FOR THE NORMALIZED LAPLACIAN

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Abstract. We provide a lower bound of the spectral radius of the universal cover of irregular graphs in the presence of symmetric edge weights. We use this bound to derive an Alon-Boppana type bound for the second eigenvalue of the normalized Laplacian.

1. Introduction

Let $G = (V, E)$ be a simply, connected, n vertex graph and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of its adjacency matrix. Letting $\lambda(G) = \lambda_2$ one of the versions of the famous Alon-Boppana theorem states that

Theorem 1 (Alon-Boppana Theorem [9]). *For any sequence of d-regular graphs G_i of increasing diameter $\liminf_{i\to\infty} \lambda(G_i) \geq 2\sqrt{d-1}$. Furthermore, for any particular d-regular graph with two edges of distance at least $2k+2$, $\lambda(G) \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{k+1}$.*

Since the spectrum of the adjacency matrix of a regular graph is closely related to the expansion properties of the graph (see [1], for example), the Alon-Boppana result may be thought of as upper bound on how good of an expander a d-regular graph can be. Recently, Friedman has confirmed a conjecture of Alon and shown that with high probability a d-regular random graph has $\lambda(G) \leq 2\sqrt{d-1}$, and thus may be thought of as extremal with respect $\lambda(\cdot)$ [3, 4].

Given the wide ranging practical and theoretical applications of expanders (see [7]), it is natural to consider what the analogue of the Alon-Boppana theorem would be for irregular graphs. To that end, say a graph has r-robust average degree d if for every vertex v, $G[V \backslash B_r(v)]$ has average degree d, where $G[S]$ is the graph induced by S and $B_r(v)$ consists of all vertices at distance most r from v. Now, with this definition Hoory generalizes the Alon-Boppana results as follows.

Theorem 2 ([6]). *Let G_i be a sequence of graphs such that G_i has r_i-robust average degree $d \geq 2$, where $r_i \to \infty$. Then $\liminf_{i\to\infty} \lambda(G_i) \geq 2\sqrt{d-1}$.*

However, in passing to the irregular case the tight relationship with the expansion of the graph is lost, and so Hoory's generalization may not be thought of as a bound on the expansion properties of a family of graphs with average degree d. In order to maintain the connection between expansion and the spectrum of an irregular graph, we consider instead the normalized Laplacian $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$ where D is the diagonal matrix of degrees. We make the standard observations that all of the eigenvalues of $\mathcal{L}$ are in $[0, 2]$ and that there is an eigenvector with eigenvalue 0, namely $\sqrt{\frac{\deg(v_i)}{\mathrm{Vol}(G)}}$, where $\mathrm{Vol}(G) = \sum_{i=1}^{n} \deg(v_i)$. Thus for a given graph G define $\lambda^{\mathcal{L}}(G)$ as the second smallest eigenvalue of the normalized Laplacian. It is well known that the $\lambda^{\mathcal{L}}(G)$ is tightly connected with expansion and algorithmic properties of G (see [2] for an overview of such results).

The natural generalization of the Alon-Boppana result to the normalized Laplacian would be that $\lambda^{\mathcal{L}}(G) \leq 1 - \frac{2\sqrt{d-1}}{d} + o(1)$ where d is the average degree of G. However, as we will show in Section 3 there exists an $\epsilon > 0$ and an infinite family of graphs $\mathcal{G}$ with common average degree d, such that $\lambda^{\mathcal{L}}(G) - \left(1 - \frac{2\sqrt{d-1}}{d}\right) > \epsilon$ for all $G \in \mathcal{G}$. Thus, our main result is that there if G_i is a sequence of graphs with average degree at least 2 and increasing robustness with respect to the average degree d and the second order average degree δ, then $\lambda^{\mathcal{L}}(G_i) \leq 1 - \frac{2d\sqrt{d-1}}{\delta} + o(1)$. We note that if the G_i's are regular then this agrees exactly with the Alon-Boppana result, however for irregular graphs this yields a higher upper bound than the natural conjecture.

2. Alon-Boppana for the Normalized Laplacian

Rather than attack the normalized Laplacian directly, we adapt the work of Hoory [6] and provide a lower bound for the spectral radius of the universal cover graph with appropriate weights. Specifically, let G be a graph with a weight function $w\colon E(G) \to \mathbb{R}^+$ and let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the universal cover of G. Noting that the weight function w lifts in a natural way to weights on the edges of $\tilde{G}$, for any $s \in \tilde{V}$ let $t_{2k}^{(w)}(s)$ be the weighted number of walks of length $2k$ from s to itself. Then the weighted spectral radius of $\tilde{G}$ is $\rho_w(\tilde{G}) = \limsup \sqrt[2k]{t_{2k}^{(w)}(v)}$ for any $v \in V(\tilde{G})$.

Theorem 3. *For any graph G with minimum degree at least 2 and weight function $w(u,v) = f(u)f(v)$, the weighted spectral radius of the universal cover is at least* $2\prod_{v\in V(G)} \left(f(v)^2\sqrt{(\deg(v)-1)}\right)^{\frac{\deg(v)}{\mathrm{Vol}(G)}}$.

Proof. We first consider the nature of the closed walks of length $2k$ starting from $v \in V(\tilde{G})$. Since the universal cover is a tree, we may view each of the steps in the walk as either a forward step or a backtracking step, according to whether it is going away from or towards v. Accordingly let T_{2k} be the set of all possible forward and backtracking sequences of a closed walk of length $2k$ and for $\tau \in T_{2k}$ let $t_{\tau,2k}^{(w)}(v)$ be the weighted number of walks of length $2k$ from v which respect the sequence τ. Notice that for any τ, there are never more backtracking steps than forward steps, and since the walk is closed the total number of forward steps is k. Thus T_{2k} corresponds to the set of Dyck paths of length $2k$ and hence the number of choices for τ is the k^{th} Catalan number, C_k.

In order to analyze these walks, it will be helpful to consider a slightly smaller class of walks where a fixed vertex is forbidden as the first step in the walk. That is, for some $v' \sim v$ we consider the set of weighted walks $\Omega_{v,v',\tau,2k}$, which are the closed length $2k$ walks starting at v, whose first step is not to v', and which respect the sequence τ. The advantage of considering these walks is that at every vertex u in the walk there are $\deg(u) - 1$ choices for a forward step in the walk. Additionally, these walks can be described by the action of a stack. Specifically, the walk starts at a vertex v with a neighbor of v, v', in the stack. Suppose the current state of the walk is that the walk is at v_i and the top element of the stack is v_i'. Then if the next step is a forward step according to τ, then a neighbor v_{i+1} of v_i other than v_i' is chosen, and the walk moves to the vertex v_{i+1} and pushes v_i onto the stack. If the next step is backwards step, then the element v_i' is popped off the stack and the walk moves to the vertex v_i'. Now, as the forward and backwards moves are governed by τ, which can be thought of as a Dyck path, the stack is always non-empty by the non-negative property of Dyck paths.

For any vertex $v' \sim v$, it is clear that $t_{\tau,2k}^{(w)}(v) \geq w\left(\Omega_{v,v',\tau,2k}\right)$, where $w(S)$ is the sum of the weights of walks in S. In fact $t_{\tau,2k}^{(w)}(v) \geq \sum_{v'\sim v} \frac{1}{\deg(v)} w\left(\Omega_{v,v',\tau,2k}\right)$. We observe that for any τ there is a natural bijection (which preserves weights) from walks on $\tilde{G}$ starting from v respecting τ and non-backtracking walks on G starting from the image of v which also respect τ. Thus, define $S_{2k}^{(w)}$ as the random variable by choosing a vertex v at random from G proportional to the degrees and then choosing a neighbor v' uniformly at random, and considering all non-backtracking walks on G starting at v with first step other than v'. That is, $S_{2k}^{(w)}$ is the number of weighted walks given a starting state (a vertex and a neighbor at the top of the stack) chosen uniformly at random. Then we have $\rho_w(\tilde{G}) = \limsup \sqrt[2k]{t_{2k}(v,w)} \geq \limsup \sqrt[2k]{\mathbb{E}\left[S_{2k}^{(w)}\right]}$. Now

$$\mathbb{E}\left[S_{2k}^{(w)}\right] = \sum_{v\in V(G)} \frac{\deg(v)}{\mathrm{Vol}(G)} \sum_{v'\sim v} \frac{1}{\deg(v)} \sum_{\tau\in T_{2k}} w\left(\Omega_{v,v',\tau,2k}\right) = \frac{1}{\mathrm{Vol}(G)} \sum_{v\in V(G)} \sum_{v'\sim v} \sum_{\tau\in T_{2k}} w\left(\Omega_{v,v',\tau,2k}\right).$$

In order to bound $\mathbb{E}\left[S_{2k}^{(w)}\right]$ below, we consider the weight of a fixed walk $\omega \in \Omega_{v,v',\tau,2k}$ as well as the probability of choosing the walk ω randomly. To that end let $(v_1,u_1),\ldots,(v_k,u_k)$ be the forward edges of the walk ω. Then define $p(\omega) = \prod_{i=1}^{k} \frac{1}{\deg(v_i)-1}$, which is the probability of choosing the walk ω at random, given that the vertex v' is not the first step. Now since each forward edge is traversed in the backtracking direction as well, the weight of the walk is $w(\omega) = \prod_{i=1}^{k} \left(f(v_i)f(u_i)\right)^2$. Thus we may rewrite

$$\mathbb{E}\left[S_{2k}^{(w)}\right] = \sum_{\tau \in T_{2k}} \sum_{v \in V(G)} \sum_{v' \sim v} \sum_{\omega \in \Omega_{v,v',\tau,2k}} \frac{p(\omega)}{\mathrm{Vol}(G)} \frac{w(\omega)}{p(\omega)}$$

and observe that $\sum_{\omega \in \Omega_{v,v',\tau,2k}} p(\omega) = 1$ and thus $\sum_{v \in V(G)} \sum_{v' \sim v} \sum_{\omega \in \Omega_{v,v',\tau,2k}} \frac{p(\omega)}{\mathrm{Vol}(G)} = 1$ and hence, by the (weighted) arithmetic-geometric mean inequality

$$\mathbb{E}\left[S_{2k}^{(w)}\right] \geq \sum_{\tau \in T_{2k}} \prod_{v \in V(G)} \prod_{v' \sim v} \prod_{\omega \in \Omega_{v,v',\tau,2k}} \left(\frac{w(\omega)}{p(\omega)}\right)^{\frac{p(\omega)}{\mathrm{Vol}(G)}}.$$

Since $\frac{w(\omega)}{p(\omega)} = \prod_{i=1}^{k} (\deg(v_i) - 1) f(v_i)^2 f(u_i)^2$, it suffices to understand for any (ordered) edge (v, v') the number of times (weighted by $\frac{p(\omega)}{\mathrm{Vol}(G)}$) any non-backtracking walk governed by τ crosses on a forward step. To that end fix the ordered edge (v, v') and let $\delta_{v,v'}(\omega, i)$ be the indicator function for the walk ω going from v to v' on the i^{th} forward step. Thus we are interested in

$$\sum_{i=1}^{k} \sum_{u \in V(G)} \sum_{u' \sim u} \sum_{\omega \in \Omega_{u,u',\tau,2k}} \frac{p(\omega)}{\mathrm{Vol}(G)} \delta_{v,v'}(\omega, i).$$

Letting $\Omega_{v,v',\tau,2k}^{(i)}$ be the prefixes in $\Omega_{v,v',\tau,2k}$ up to the i^{th} forward step and letting $p^{(i)}(\omega^{(i)}) = \prod_{j=1}^{i} \frac{1}{\deg(v_j)-1}$, we have that

$$\sum_{i=1}^{k} \sum_{u \in V(G)} \sum_{u' \sim u} \sum_{\omega \in \Omega_{u,u',\tau,2k}} \frac{p(\omega)}{\mathrm{Vol}(G)} \delta_{v,v'}(\omega, i) = \sum_{i=1}^{k} \sum_{u \in V(G)} \sum_{u' \sim u} \sum_{\omega^{(i)} \in \Omega_{u,u',\tau,2k}^{(i)}} \frac{p^{(i)}(\omega^{(i)})}{\mathrm{Vol}(G)} \delta_{v,v'}(\omega^{(i)}, i).$$

But then observing that the choice of $p^{(i)}(\cdot)$ defines a Markov chain on the states of the random walk and further defines a uniform transition among neighboring states. A consequence, $\frac{1}{\mathrm{Vol}(G)}$ is the limiting distribution of the Markov chain, and hence

$$\begin{aligned}
\sum_{i=1}^{k} \sum_{u \in V(G)} \sum_{u' \sim u} \sum_{\omega^{(i)} \in \Omega_{u,u',\tau,2k}^{(i)}} \frac{p^{(i)}(\omega^{(i)})}{\mathrm{Vol}(G)} \delta_{v,v'}(\omega^{(i)}, i) &= k \sum_{u \in V(G)} \sum_{u' \sim u} \sum_{\omega^{(1)} \in \Omega_{u,u',\tau,2k}^{(1)}} \frac{p^{(1)}(\omega^{(1)})}{\mathrm{Vol}(G)} \delta_{v,v'}(\omega^{(i)}, 1) \\
&= k \sum_{\substack{u' \sim v \\ u' \neq v'}} \frac{1}{\mathrm{Vol}(G)\,(\deg(v) - 1)} \\
&= \frac{k}{\mathrm{Vol}(G)}.
\end{aligned}$$

Thus we have that

$$\mathbb{E}\left[S_{2k}^{(w)}\right] \geq \sum_{\tau \in T_{2k}} \prod_{v \in V(G)} \prod_{v' \sim v} \left(f(v)^2 f(v')^2 (\deg(v) - 1)\right)^{\frac{k}{\mathrm{Vol}(G)}} = C_k \prod_{v \in V(G)} \left((\deg(v) - 1) f(v)^4\right)^{\frac{\deg(v)k}{\mathrm{Vol}(G)}}.$$

This proves the result as

$$\rho_w(\tilde{G}) \geq \limsup_{k \to \infty} \sqrt[2k]{\mathbb{E}\left[S_{2k}^{(w)}\right]} \geq 2 \prod_{v \in V(G)} \left(f(v)^2 \sqrt{(\deg(v) - 1)}\right)^{\frac{\deg(v)}{\mathrm{Vol}(G)}}$$

□

Corollary 4. *For any graph G with minimum degree at least 2 with weight function $w(u, v) = (\deg(v)\deg(u))^{-1/2}$, the weighted spectral radius of the universal cover is at least $2\sqrt{\prod_{v \in V(G)} \left(\frac{\deg(v)-1}{\deg(v)^2}\right)^{\frac{\deg(v)}{\mathrm{Vol}(G)}}}$.*

Now letting d be the average degree and δ be the second moment of the degree sequence (that is, $\delta = \frac{1}{n}\sum_{v\in V(G)} \deg(v)^2$, we can reformulate this bound into a more natural one in terms of global statistics of G. Specifically, since $(x-1)^x$ is log-convex for $x \geq 2$, $\prod_{v\in V(G)}(\deg(v)-1)^{\frac{\deg(v)}{\mathrm{Vol}(G)}} \geq (d-1)^{\frac{dn}{\mathrm{Vol}(G)}} = d-1$. Additionally, by the arithmetic-geometric mean inequality,

$$\prod_{v\in V(G)} \deg(v)^{\frac{\deg(v)}{\mathrm{Vol}(G)}} \leq \frac{\sum_{v\in V(G)}\deg(v)^2}{\mathrm{Vol}(G)} = \frac{\sum_{v\in V(G)}\deg(v)^2}{dn} = \frac{\delta}{d}.$$

Hence we have if G is a graph with minimum degree at least 2, then $\rho_w(G) \geq \frac{2d\sqrt{d-1}}{\delta}$. Building on this observation we have the following natural extension.

Theorem 5. *If G is a graph with average degree at least 2 and $w(\{u,v\}) = (\deg(u)\deg(v))^{-1/2}$, then $\rho_w(\tilde{G}) \geq \frac{2d\sqrt{d-1}}{\delta}$.*

Proof. Since the average degree of G is at least 2 and removing a degree one vertex can only increase the average degree, G has a non-empty 2-core, G'. By adapting the proof of Theorem 3, we have that

$$\rho_w(\tilde{G}) \geq \rho_w(\tilde{G}') \geq 2\sqrt{\prod_{v'\in V(G')}\left(\frac{\deg_{G'}(v')-1}{\deg_G(v')^2}\right)^{\frac{\deg_{G'}(v')}{\mathrm{Vol}(G')}}}.$$

Note that the first inequality comes from the limiting of the closed walks to those entirely within G' while preserving the weight of all those walks. Now since G' has minimum degree at least 2 by definition and deleting degree one vertices only increases the average degree,

$$\prod_{v'\in V(G')}(\deg_{G'}(v')-1)^{\frac{\deg_{G'}(v')}{\mathrm{Vol}(G')}} \geq \bar{d}'-1 \geq d-1.$$

We observe that if $x \geq 2y$ and $\alpha \in (0,1]$, then $\alpha^{\frac{x}{y}} \geq \alpha^{\frac{x+1}{y+2}}$. Thus by sequentially adding the vertices deleted to reach the two core, we have

$$\prod_{v'\in V(G')}\deg_G(v')^{-\frac{\deg_{G'}(v')}{\mathrm{Vol}(G')}} = \prod_{v\in V(G)}\deg_G(v)^{-\frac{\deg_{G'}(v)}{\mathrm{Vol}(G')}} \geq \prod_{v\in V(G)}\deg_G(v)^{-\frac{\deg_G(v)}{\mathrm{Vol}(G)}} \geq \frac{d}{\delta}.$$

Combining these observations gives the desired result. ☐

Now define $B_r(v)$ as the set of vertices that are distance at most r from v and recall that a graph G has radius at least k if for every vertex $v \in V(G)$ there is a vertex u so that the distance between u and v is at least k. Alternatively, we may say that a graph had radius of at least k if for every $v \in V(G)$, $B_{k-1}(v) \subsetneq B_k(v)$. Additionally, for a set $S \subseteq V(G)$ we define $G[S]$ as the subgraph of G induced by S. Following the notation of Hoory, we will say that a graph G has r-robust second order degrees (d,δ) if for every vertex $v \in V(G)$, the graph $G[V(G)\backslash B_r(v)]$ has average degree at least d and the average of $\deg_G(w)^2$ for $w \in V(G)\backslash B_r(v)$ is at most δ. Using this notation, we now have the following analogue of the Alon-Boppana result.

Theorem 6. *If G has r-robust second order average degrees (d,δ) with $d \geq 2$, then $\lambda^{\mathcal{L}}(G) \leq 1-\frac{2d\sqrt{d-1}}{\delta}\left(1-c\frac{\log(r)}{r}\right)$ for some constant c.*

Proof. Let $M = I-\mathcal{L}$, $2 \leq 2k \leq r$ and $w(u,v) = (\deg(u)\deg(v))^{-1/2}$. Fix an arbitrary vertex x and let $V_x = V\backslash B_r(x)$ and $G_x = G[V_x]$. Now by Theorem 3,

$$\rho_w(\tilde{G}_x) \geq 2\prod_{v\in V_x}\left(\frac{\sqrt{\deg_{G_x}(v)-1}}{\deg_G(v)}\right)^{\frac{\deg_{G_x}(v)}{\mathrm{Vol}(G_x)}}.$$

As in the proof of Theorem 5,

$$\prod_{v\in V_x}\deg_{G_x}(v)^{\frac{\deg_{G_x}(v)}{\mathrm{Vol}(G_x)}} \geq d-1$$

and

$$\prod_{v \in V_x} \deg_G(v)^{\frac{\deg_{G_x}(v)}{\mathrm{Vol}(G_x)}} \leq \sum_{v \in V_x} \frac{\deg_G(v)\deg_{G_x}(v)}{\mathrm{Vol}(G_x)} \leq \sum_{v \in V_x} \frac{\deg_G(v)^2}{d\,|V_x|} = \frac{\delta}{d}.$$

Thus, there is some vertex $v \in V(G_x)$ such that $t_{2k}^{(w)}(v) \geq C_k\left(\frac{d^{2k}(d-1)^k}{\delta^{2k}}\right)$. Now letting $V_v = V \backslash B_r(v)$ and $G_v = B[V_v]$, there is some vertex $u \in V_v$ such that $t_{2k}^{(w)}(u) \geq C_k\left(\frac{d^{2k}(d-1)^k}{\delta^{2k}}\right)$. Thus we can define the vector $f = \frac{1}{\sqrt{\deg_G(v)+\deg_G(u)}}\left(\sqrt{\deg_G(u)}\mathbb{1}_v - \sqrt{\deg_G(v)}\mathbb{1}_u\right)$ and observe that

$$\begin{aligned} f^T M^{2k} f &= \frac{\deg_G(v)\mathbb{1}_u^T M^{2k}\mathbb{1}_u - 2\sqrt{\deg_G(u)\deg_G(v)}\mathbb{1}_u^T M^{2k}\mathbb{1}_v + \deg_G(u)\mathbb{1}_v^T M^{2k}\mathbb{1}_u}{\deg_G(u)+\deg_G(v)} \\ &\geq \frac{(\deg_G(u)+\deg_G(v))\,C_k\left(\frac{d^{2k}(d-1)^k}{\delta^{2k}}\right) - 2\sqrt{\deg_G(u)\deg_G(v)}\mathbb{1}_u^T M^{2k}\mathbb{1}_v}{\deg_G(u)+\deg_G(v)} \\ &= C_k\left(\frac{d^{2k}(d-1)^k}{\delta^{2k}}\right), \end{aligned}$$

where the final inequality comes from the observation that the distance between u and v is at least $r+1 > 2k$. Observing that f is orthogonal to the principal eigenvalue of M and taking the $2k^{\text{th}}$ root yields the result. $\square$

Corollary 7. *Let G_n be a sequence of n vertex graphs with average degrees $d_n \geq 2$ and second order average degrees δ_n and such that the maximum degrees Δ_n are $n^{o(1)}$. If $\lim_{n\to\infty} 1 - \frac{2d_n\sqrt{d_n-1}}{\delta_n} = L$, then $\limsup_{n\to\infty}\lambda^{\mathcal{L}}(G_n) = L$.*

Proof. This follows immediately from Theorem 6 by noting that by the choice of maximum degrees there is an increasing sequence r_n such that each G_n has r_n-robust second order average degrees $(d_n(1-o(1)), \delta_n(1+o(1))$. $\square$

We note that in general the natural conjectured bound on $\lambda^{\mathcal{L}}(G)$ extending Alon-Boppana result is $1 - \frac{2\sqrt{d-1}}{d}$ which is in general smaller than $1 - \frac{2d\sqrt{d-1}}{\delta}$. In the following section, we show that this separation is essential by providing a class of graphs such that $\lambda^{\mathcal{L}}(G) \geq 1 - 2\frac{\sqrt{d-1}}{d} + \epsilon$ for some fixed positive ϵ.

3. REGULAR GRAPHS ARE NOT EXTREMAL

We first observe that there is a trivial obstruction to regular graphs being extremal with respect to $\lambda^{\mathcal{L}}$. Specifically, if G_n is a sequence of d-regular nearly Ramanujan n-vertex graph, then the graphs G'_n formed by adding a dominating vertex have average degree approaching $d+2$, while $\limsup_{n\to\infty}\lambda^{\mathcal{L}}(G'_n) = 1 - \frac{2\sqrt{d-1}}{d+1} > 1 - \frac{2\sqrt{d+1}}{d+2}$. However, all the graphs G'_n have diameter 2, in contrast to the proof of Nilli which uses the diameter to control the error term [9]. Thus, one might suppose that it suffices to impose a growing diameter condition to recover the natural generalization of Alon-Boppana. However, in this section we will provide a means of constructing an infinite family of graphs $\{G_i\}$, with common average degree d and common maximal degree (and hence increasing diameter), such that $\liminf_{i\to\infty}\lambda^{\mathcal{L}}(G) \geq 1 - \frac{2\sqrt{d-1}}{d} + \epsilon$ for some fixed $\epsilon > 0$. To that end, given graphs H_1 on n_1 vertices, H_2 on n_2 vertices, and B a bipartite graph on n_1, n_2) vertices, we define $G(H_1, H_2, B)$, in the natural way, gluing the vertices of H_1 and H_2 to the appropriate side of the bipartition of B.

Lemma 8. *If H_1 is an n vertex d_1-regular graph, H_2 is a rn vertex d_2-regular graph, and B is a (n, rn) vertex (rk, k)-regular bipartite graph, then $G = G(H_1, H_2, B)$ is such that*

$$\max\left\{\omega, \frac{\lambda(G_1)}{d_1+rk}, \frac{\lambda(G_2)}{d_2+k}\right\} \leq 1 - \lambda^{\mathcal{L}}(G) \leq \max\{\omega, \rho\}$$

where

$$\rho = \begin{cases} \frac{\frac{1}{4}\lambda(B)^2 - \lambda(G_1)\lambda(G_2)}{\sqrt{(d_1+rk)(d_2+k)}\lambda(B) - (d_2+k)\lambda(G_1) - (d_1+rk)\lambda(G_2)} & \frac{\lambda(G_1)}{d_1+rk} + \frac{\lambda(G_2)}{d_2+k} < \frac{\lambda(B)}{\sqrt{(d_1+rk)(d_2+k)}} \\ \max\left\{\frac{\lambda(G_1)}{d_1+rk}, \frac{\lambda(G_2)}{d_2+k}\right\} & \frac{\lambda(G_1)}{d_1+rk} + \frac{\lambda(G_2)}{d_2+k} \geq \frac{\lambda(B)}{\sqrt{(d_1+rk)(d_2+k)}} \end{cases}$$

and

$$\omega = \frac{1}{d}\frac{r}{r+1}\left(\frac{(d_2+k)d_1}{d_1+rk} - 2k + \frac{(d_1+rk)d_2}{(d_2+k)r^2}\right).$$

Proof. Rather than dealing directly with the normalized Laplacian, $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$, we will again deal with the matrix $M = D^{-1/2}AD^{-1/2}$. Now the largest eigenvalue of M has value one (corresponding to the zero eigenvalue of $\mathcal{L}$) and has eigenvector $D^{1/2}\mathbb{1}$. For convenience of notation, let $\mathbb{1}_t$ be an appropriately sized vector whose first t entries are 1 and remaining entries are zero, and similarly let $\mathbb{1}'_s$ be an appropriately sized vector whose last s entries are one and the remaining entries are zero. We fix an ordering of vertices of G so that the n side of the bipartition appears first and thus the primary eigenvector of M is $\sqrt{d_1+rk}\mathbb{1}_n + \sqrt{d_2+k}\mathbb{1}'_{rn}$.

Let τ be the unit vector

$$\sqrt{\frac{(d_2+k)r}{\operatorname{Vol}(G)}}\mathbb{1}_n - \sqrt{\frac{d_1+rk}{r\operatorname{Vol}(G)}}\mathbb{1}'_{rn}.$$

Now any unit vector v orthogonal to the first eigenspace of M, can be written in the form $\alpha f + \beta g + \gamma\tau$ where $\alpha^2+\beta^2+\gamma^2 = 1$, $\|f\| = \|g\| = 1$, $f^T\tau = g^T\tau = f^Tg = 0$, and f is only non-zero on the first n entries and g is only non-zero on the last rn entries. Thus to understand v^TMv it suffices to understand f^TMf, g^TMg, f^TMg, $f^TM\tau$, $g^TM\tau$, and $\tau^TM\tau$. The lower bound comes immediately from considering f^TMf, g^TMg, and $\tau^TM\tau$ and the value of $\tau^TM\tau$ which we calculate later.

It is easy to see that $f^TMf \leq \frac{\lambda(H_1)}{d_1+rk}$ and $g^TMg \leq \frac{\lambda(H_2)}{d_2+k}$. Noting that $M\tau = \eta\mathbb{1}_n + \zeta\mathbb{1}'_{rn}$ for some η and ζ, it is clear that $f^TM\tau = g^TM\tau = 0$. Observing that

$$\begin{aligned}
\eta &= \frac{d_1}{d_1+rk}\sqrt{\frac{(d_2+k)r}{\operatorname{Vol}(G)}} - \frac{rk}{\sqrt{(d_1+rk)(d_2+k)}}\sqrt{\frac{d_1+rk}{r\operatorname{Vol}(G)}}\\
\zeta &= \frac{k}{\sqrt{(d_1+rk)(d_2+k)}}\sqrt{\frac{(d_2+k)r}{\operatorname{Vol}(G)}} - \frac{d_2}{d_2+k}\sqrt{\frac{d_1+rk}{r\operatorname{Vol}(G)}},
\end{aligned}$$

we then have

$$\begin{aligned}
\tau^TM\tau &= \eta n\sqrt{\frac{(d_2+k)r}{\operatorname{Vol}(G)}} - \zeta rn\sqrt{\frac{d_1+rk}{r\operatorname{Vol}(G)}}\\
&= \frac{d_2+k}{d_1+rk}\frac{d_1rn}{\operatorname{Vol}(G)} - \frac{krn}{\operatorname{Vol}(G)} - \frac{krn}{\operatorname{Vol}(G)} + \frac{d_1+rk}{d_2+k}\frac{d_2n}{r\operatorname{Vol}(G)}\\
&= \frac{1}{d}\frac{r}{r+1}\left(\frac{(d_2+k)d_1}{d_1+rk} - 2k + \frac{(d_1+rk)d_2}{(d_2+k)r^2}\right)
\end{aligned}$$

We now consider f^TMg. If we let u be the concatenation of the vectors f and g, then $f^TMg + g^TMf = \frac{1}{\sqrt{(d_1+rk)(d_2+k)}}u^TA_Bu$, where A_B is the adjacency matrix for the graph B. Furthermore, the vector u formed in this manner spans a $(r+1)n-2$ dimensional subspace of vectors and the orthogonal complement is spanned by $\sqrt{rk}\mathbb{1}_n + \sqrt{k}\mathbb{1}'_{rn}$ and $\sqrt{rk}\mathbb{1}_n - \sqrt{k}\mathbb{1}'_{rn}$. But since $\sqrt{rk}\mathbb{1}_n + \sqrt{k}\mathbb{1}'_{rn}$ is the principle eigenvector for A_B and $\left(\sqrt{rk}\mathbb{1}_n - \sqrt{k}\mathbb{1}'_{rn}\right)^T A_B\left(\sqrt{rk}\mathbb{1}_n - \sqrt{k}\mathbb{1}'_{rn}\right) = -2\sqrt{k}rkn < 0$, $u^TA_Bu \leq \lambda(B)$. Thus $f^TMg \leq \frac{\lambda(B)}{2\sqrt{(d_1+rk)(d_2+k)}}$.

Now since $f^TM\tau = g^TM\tau$, the second largest eigenvalue of M occurs either when $\gamma^2 = 0$ or when $\alpha^2+\beta^2 = 0$. Thus, optimizing for choice of α when $\gamma = 0$, gives the result. ☐

Corollary 9. *If H_1 is an n vertex d_1-regular Ramanujan graph, H_2 is a rn vertex d_2-regular Ramanujan graph, and B is a (n, rn) vertex (rk, k)-regular bipartite Ramanujan graph, if $3d_2 > k+4$ and r is sufficiently large, then $G = G(H_1, H_2, B)$ is such that $\lambda^{\mathcal{L}}(G) = 1 - \frac{2\sqrt{d_2-1}}{d_2+k}$.*

Proof. For large enough r, $\omega < 0$. Additionally, $\frac{2\sqrt{d_1-1}}{d_1+rk} \to 0$ and $\frac{\sqrt{k-1}+\sqrt{rk-1}}{\sqrt{(d_1+rk)(d_2+k)}} \to \frac{1}{\sqrt{d_2+k}}$ as $r \to \infty$. Now since $3d_2 > k+4$, it follows that $\frac{2\sqrt{d_2-1}}{d_2+k} > \frac{1}{\sqrt{d_2+k}}$ and by Lemma 8 the result follows. ☐

In fact, it suffices for that graphs H_1, H_2, and B the nearly Ramanujan graphs. That is, for $\lambda(H_1) \leq 2\sqrt{d_1 - 1} + o(1)$, $\lambda(H_2) \leq 2\sqrt{d_2 - 1} + o(1)$, and $\lambda(B) \leq \sqrt{k-1} + \sqrt{rk-1} + o(1)$ where the $o(1)$ is in terms of n.

Theorem 10. *For any fixed choice of integers* $d_1 \geq 3, d_2 \geq 8$, *there is an infinite family of graphs* $\{G_i\}_{i \in I}$ *with common average degree* d, *such that* $\lambda^{\mathcal{L}}(G_i) \geq 1 - \frac{2\sqrt{d-1}}{d} + \epsilon$ *for some fixed* $\epsilon > 0$.

Proof. First we observe that since $\frac{(d_2+3)d_1}{d_1+3r} + \frac{(d_1+3r)d_2}{(d_2+3)r^2} \to 0$ as $r \to \infty$ and there is a choice of r so that $\frac{(d_2+3)d_1}{d_1+3r} + \frac{(d_1+3r)d_2}{(d_2+3)r^2} < 6$ and $d_1 < (r+1)(d_2+6)$. Now let $\{H_n\}$ be a sequence of d_1-regular nearly Ramanujan graphs on n vertices, and let $\{\hat{H}_n\}$ be a sequence of d_2-regular nearly Ramanujan graphs on rn vertices, and let $\{B_n\}$ be a sequence of random $(3r, 3)$-regular bipartite graphs on (n, rn) vertices. We note that by the work of Friedman [4], the classes $\{H_n\}$ and $\{\hat{H}_n\}$ exist. Define $G_n = G(H_n, \hat{H}_n, B_n)$. Now the average degree for each G_n is $\frac{r}{r+1}(d_2+6) + \frac{d_1}{r+1} < d_2 + 6$ by the choice of r. Furthermore, the choice of r and the observation that $\frac{2\sqrt{d_2-1}}{d_2+3} > \frac{3\sqrt{r}}{\sqrt{(d_1+3r)(d_2+3)}}$, together with Corollary 9, gives that $1 - \lambda^{\mathcal{L}}(G_n) = \frac{2\sqrt{d_2-1}+o(1)}{d_2+3}$. Since $\frac{2\sqrt{x-1}}{x}$ is a decreasing function for $x \geq 2$ and $d \leq d_2 + 6$, it suffices to show that there is an $\epsilon > 0$ such that $\frac{2\sqrt{d_2-1}}{d_2+3} < \frac{2\sqrt{d_2+5}}{d_2+6}$. Rearranging, this is equivalent to $\frac{d_2+6}{d_2+3} < \sqrt{\frac{d_2+5}{d_2-1}}$. Since both sides are positive, it suffices to show that $1 + \frac{6}{d_2+3} + \frac{9}{(d_2+3)^2} < 1 + \frac{6}{d_2-1}$. Alternatively we may show that $6d_2^2 + 21d_2 - 27 = (6(d_2+3)+9)(d_2-1) < 6(d_2+3)^2 = 6d_2^2 + 36d_2 + 54$, which clearly holds. Thus there is an $\epsilon > 0$ such that for a sufficiently large n, $\lambda^{\mathcal{L}}(G_n) \geq 1 - \frac{2\sqrt{d-1}}{d} + \epsilon$. ☐

It is worth noting that this construction could be extended to larger class of degrees if the existence of a larger class of nearly Ramanujan biregular bipartite graphs were known. Although it is clear that by subdivision any k-regular nearly Ramanujan graph gives rise to a $(2, k)$-regular nearly Ramanujan bipartite graph [5] and Li and Solé have provided a construction of a limited class of biregular bipartite graphs based generalized n-gons [8], neither of these constructions yields a sufficient diversity of bipartite near Ramanujan graphs to meaningfully expand the range of degrees chosen.

4. Spectral Bounds for the Normalized Laplacian Spectral of Bipartite Graphs

In order to deal with bipartite graphs, we need the following result from Hoory [6]. First, let T_{2k} be the collection of length $2k$ Dyck paths and for $\tau \in T_{2k}$ let $\mathrm{odd}(\tau)$ be the number of positive steps on τ starting from an odd height. Similarly, define $\mathrm{even}(\tau)$ and note that $\mathrm{odd}(\tau) + \mathrm{even}(\tau) = k$.

Lemma 11. *For any positive constants* $a, b > 0$,

$$\lim_{k \to \infty} \sqrt[2k]{\sum_{\tau \in T_{2k}} a^{\mathrm{even}(\tau)} b^{\mathrm{odd}(\tau)}} = \sqrt{a} + \sqrt{b}.$$

For the sake of completeness we provide this alternative proof.

Proof. Let $C_k^{(a,b)} = \sum_{\tau \in T_{2k}} a^{\mathrm{even}(\tau)} b^{\mathrm{odd}(\tau)}$ and let $C(a, b, x) = \sum_{k=0}^{\infty} C_k^{(a,b)} x^k$. Making the standard observation that for any Dyck path of length $2k$ which first returns to height 0 at step $2t$, the subpath from step 1 to step $2t-1$ is also a Dyck path, we have that $C_{k+1}^{(a,b)} = \sum_{i=0}^{k} a C_i^{(b,a)} C_{k-i}^{(a,b)}$, where the interchange in (a, b) occurs because the sub-Dyck path starts at an odd value. Thus we have that $C(a, b, x) = 1 + axC(b, a, x)C(a, b, x)$ and $C(b, a, x) = 1 + bxC(a, b, x)C(b, a, x)$. Letting $C^*(\{a, b\}, x) = C(a, b, x)C(b, a, x)$ and combining these relationships we get $C^* = 1 + (a+b)xC^* + abx^2C^{*2}$. Thus

$$C^*(\{a, b\}, x) = \frac{1 - (a+b)x - \sqrt{(1-(a+b)x)^2 - 4abx^2}}{2abx^2},$$

where the negative square root is chosen to eliminate the pole at $x = 0$. Thus

$$C(a, b, x) = 1 + ax\frac{1 - (a+b)x - \sqrt{(1-(a+b)x)^2 - 4abx^2}}{2abx^2} = \frac{1 + (b-a)x - \sqrt{1 - 2(a+b)x + (b-a)^2x^2}}{2bx}.$$

Note that this has poles at $x = \frac{1}{(\sqrt{a}+\sqrt{b})^2}$ and $x = \frac{1}{(\sqrt{a}-\sqrt{b})^2}$ (if $a \neq b$), and thus $C_k^{(a,b)} \sim \left(\sqrt{a}+\sqrt{b}\right)^{2k}$ as desired. □

With this lemma in hand, we now have the following normalized Laplacian analogue of Hoory's bound on the spectral radios of the universal cover of irregular bipartite graphs [6].

Theorem 12. *For any bipartite graph $B = (L, R, E)$ with minimum degree at least 2 and weight function $w(u, v) = f(u)f(v)$, the weighted spectral radius of the universal cover is at least*

$$\left(\sqrt{d_L - 1} + \sqrt{d_R - 1}\right) \sqrt{\prod_{v \in L \cup R} f(v)^{\frac{2\deg(v)}{|E|}}},$$

where d_L and d_R are the average degrees of the L and R sides of the partition, respectively.

Proof. We will consider the same class of walks as in the proof of Theorem 3, except the starting vertex will be restricted to vertices in L. Specifically, using the same notation, we have

$$\rho_w(\tilde{B}) \geq \limsup_{k\to\infty} \sqrt[2k]{\sum_{v\in L} \frac{\deg(v)}{|E|} \sum_{v'\sim v} \frac{1}{\deg(v)} \sum_{\tau\in T_{2k}} w(\Omega_{v,v',\tau,2k})} = \limsup_{k\to\infty} \sqrt[2k]{\frac{1}{|E|} \sum_{v\in L} \sum_{v'\sim v} \sum_{\tau\in T_{2k}} w(\Omega_{v,v',\tau,2k})}.$$

Thus we consider

$$\sum_{\tau\in T_{2k}} \sum_{v\in L} \sum_{v'\sim v} \sum_{\omega\in\Omega_{v,v',\tau,2k}} \frac{w(\omega)}{|E|} \frac{p(\omega)}{p(\omega)} \geq \sum_{\tau\in T_{2k}} \prod_{v\in L} \prod_{v'\sim v} \prod_{\omega\in\Omega_{v,v',\tau,2k}} \left(\frac{w(\omega)}{p(\omega)}\right)^{\frac{p(\omega)}{|E|}}.$$

Now since $\frac{w(\omega)}{p(\omega)} = \prod_{i=1}^{k} (\deg(v_i) - 1) f(v_i)^2 f(u_i)^2$ it suffices to understand for any edge how many time the ordered edge (v, v') is crossed in a non-backtracking walk by a forward step (weighted by $\frac{p(\omega)}{|E|}$). That is, we are interested in

$$\sum_{i=1}^{k} \sum_{u\in L} \sum_{u'\sim u} \sum_{\omega\in\Omega_{u,u',\tau,2k}} \delta_{v,v'}(\omega, i).$$

If $v \in L$, this is $\mathrm{even}(\tau)$ while if $v \in R$, this is $\mathrm{odd}(\tau)$. Thus

$$\begin{aligned} \sum_{v\in L} \sum_{v'\sim v} \sum_{\omega\in\Omega_{v,v',\tau,2k}} \frac{w(\omega)}{|E|} \frac{p(\omega)}{p(\omega)} &= \prod_{v\in L} (\deg(v)-1)^{\frac{\deg(v)\,\mathrm{even}(\tau)}{|E|}} \prod_{v\in R} (\deg(v)-1)^{\frac{\deg(v)\,\mathrm{odd}(\tau)}{|E|}} \prod_{v\in L\cup R} f(v)^{\frac{4\deg(v)k}{|E|}} \\ &\geq (d_L - 1)^{\ \mathrm{odd}(\tau)} (d_R - 1)^{\ \mathrm{even}(\tau)} \prod_{v\in L\cup R} f(v)^{\frac{2\deg(v)(\mathrm{even}(\tau)+\mathrm{odd}(\tau))}{|E|}}. \end{aligned}$$

Thus $\rho_w(\tilde{B}) \geq \left(\sqrt{d_L - 1} + \sqrt{d_R - 1}\right) \sqrt{\prod_{v\in L\cup R} f(v)^{\frac{2\deg(v)}{|E|}}}$. □

Applying the weighting from the normalized Laplacian, we have the following result.

Corollary 13. *For any bipartite graph $B = (L, R, E)$ with minimum degree at least 2 and weight function $w(u, v) = \frac{1}{\sqrt{\deg(v)\deg(u)}}$, the weighted spectral radius of the universal cover is at least*

$$\left(\sqrt{d_L - 1} + \sqrt{d_R - 1}\right) \sqrt{\frac{d_L d_R}{\delta_L \delta_R}},$$

where d_L and d_R are the average degrees of the sides and δ_L and δ_R are the second order average degrees of the sides.

It is worth noting that the term $\sqrt{\frac{d_R d_L}{\delta_R \delta_L}}$ can be replaced by $\frac{d_R d_L}{\hat{d}_R \hat{d}_L}$ where $\hat{d}_R$ and $\hat{d}_L$ are the averages of the $3/2$ powers of the degrees in R and L, respectively. For a bipartite graph B with bipartition (L, R), define $\kappa'(B) = \left(\sqrt{d_L - 1} + \sqrt{d_R - 1}\right) \frac{d_L d_R}{\hat{d}_L \hat{d}_R}$.

It is also worth noting that unlike Theorem 3, this theorem can not be extend to the case where the average degree is at least 2, as the average degree on each side of the partition could decreased by deleting a vertex of degree 1.

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