Math 2602 Exam #3

Fall 2008

Name: ______________________________

GTid (9xxxxxxxx): ______________________________
Instructor: Stephen J. Young

There are 4 questions on this exam on 4 pages (not counting this coverpage).

- Be sure to fully explain your answers, as answers that are not accompanied by explanations/work may receive no credit.
- Place your name and problem number on each solution sheet. The exams will be separated to be graded. Anyone turning in a solution sheet without a name will receive stern looks and runs the risk of the scores not being accurately totalled.
- You are to complete this exam completely alone, without the aid of calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.

By signing on the line below, you agree to abide by the Georgia Tech Honor Code, the principles of which are embodied by the Challenge Statement:

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community.

Student signature: ______________________________

Question:	1	2	3	4	Total
Points:	5	5	5	5	20
Score:					

1. (5 points) Translate the following linear program into standard form and find the optimal solution using the simplex algorithm.

$$\begin{aligned} \max \quad & \overset{3x_1+x_2}{\cancel{3x_1 + x_2}} \\ \text{subject to} \quad & \\ -2x_1 + x_2 &\le 2 \\ x_1 + x_2 &\le 6 \\ x_1 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

The standard form for this LP is

$$\begin{aligned} &\max \ 3x_1+x_2 \\ &\text{s.t.} \quad -2x_1+x_2+s_1 = 2 \\ &\qquad x_1+x_2+s_2=6 \\ &\qquad x_1 \qquad +s_3=4 \\ &\qquad x_1, x_2, s_1, s_2, s_3 \ge 0 \end{aligned}$$

This yields an initial simplex tableau of

$$\begin{array}{r c} & \begin{bmatrix} -3 & -1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \\ \begin{matrix} \\ \\ 6/1=6 \\ 4/1=4 \ \text{smallest ratio} \rightarrow \end{matrix} & \\ & \uparrow \text{ most negative entry} \end{array}$$

2^{nd} tableau

$$\begin{array}{r c} \text{smallest ratio} \rightarrow \text{(row 3)} & \begin{bmatrix} 0 & -1 & 0 & 0 & 3 & 12 \\ 0 & 1 & 1 & 0 & 2 & 10 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \\ & \uparrow \text{ most negative} \end{array}$$

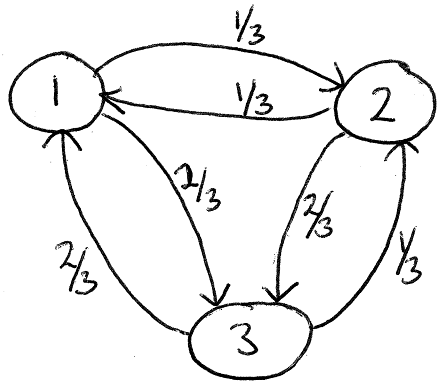
3^{rd} tableau

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 14 \\ 0 & 0 & 1 & -1 & 3 & 8 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

No negative entries hence optimal

The optimal solution is $(4,2,8,0,0)$ with value 14.

3. (5 points) The following Markov Chain has a unique limiting distribution, find it and explain why it is unique.



The matrix representing the Makov Chain is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1/3 & 2/3 \\ 2 & 1/3 & 0 & 1/3 \\ 3 & 2/3 & 2/3 & 0 \end{array} = M$$

$$\det(M-\lambda I) = \det\begin{pmatrix} -\lambda & 1/3 & 2/3 \\ 1/3 & -\lambda & 1/3 \\ 2/3 & 2/3 & -\lambda \end{pmatrix} = -\lambda^3 + 2/27 + 4/27 + \frac{4}{9}\lambda + \frac{2}{9}\lambda + \frac{1}{9}\lambda$$

$$= -\lambda^3 + \frac{7}{9}\lambda + \frac{2}{9} = (1-\lambda)(\lambda^2+\lambda+2/9)$$

Since we know $\lambda=1$ is a root,

$$\begin{array}{r|l} & \lambda^2+\lambda+2/9 \\ \hline 1-\lambda & -\lambda^3+0\lambda^2+\frac{7}{9}\lambda+\frac{2}{9} \\ & -\lambda^3+\lambda^2 \\ \hline & -\lambda^2 \\ & -\lambda^2+\lambda \\ \hline & -\frac{2}{9}\lambda+2/9 \end{array}$$

Since 1 is not a root of $\lambda^2+\lambda+2/9$, we know the limiting distribution is unique since 1 has multiplicity one in the original characteristic polynomial

$$\left[\begin{array}{ccc|c} -1 & 1/3 & 2/3 & 0 \\ 1/3 & -1 & 1/3 & 0 \\ 2/3 & 2/3 & -1 & 0 \end{array}\right] \xrightarrow[\times 3]{\text{all rows}} \left[\begin{array}{ccc|c} -3 & 1 & 2 & 0 \\ 1 & -3 & 1 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right] \xrightarrow[r_1+3r_2]{r_3-2r_2} \left[\begin{array}{ccc|c} 0 & -8 & 5 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 8 & 5 & 0 \end{array}\right]$$

Thus let $v_2=5$
$v_3=8$

Then $v_1-15+8=0$
so $v_1=7$

Thus the eigenvector for $\lambda=1$ is $\begin{bmatrix}7\\5\\8\end{bmatrix}$ normalizing to get a distribution we get $\begin{bmatrix}7/20\\1/4\\2/5\end{bmatrix}$ as the limiting distribution.

2. (5 points) Write an linear program to solve the following problem and translate it into canonical form. Be sure to explain what each variable in your initial formulation represents.

A burger joint wants to create a new healthier burger by mixing beef, chicken, and a soy vegetable mixture. In each gram of beef there are 2.6 calories, 0.2 grams of fat, 1 mg of cholesterol, and 1 mg of sodium. Each gram of chicken has 1.5 calories, 0.07 grams of fat, 0.8 mg of cholesterol, and 0.8 mg of sodium. The vegetable mixture contains 2.0 calories, 0.1 grams of fat, no cholesterol, and 4 mg of sodium. The resulting burger must weight at least 150 grams, have no more than 375 calories, no more than 10 grams of fat, no more than 50 mg of cholesterol, and no more than 400 mg of sodium. The beef costs half a cent per gram, the chicken costs .07 cents per gram, and the vegetable mixture costs a cent per gram. Find the mixture that satisfies all the requirements and is the cheapest.

	cal/g	fat/g	chol/g	sodium/g	¢/g
beef	2.6	.2	1	1	.5
chicken	1.5	.07	.8	.8	.07
veggie	2.0	.1	0	4	1

$b \equiv$ grams of beef
$c \equiv$ grams of chicken
$v \equiv$ grams of veggie

min $.5b + .07c + v$

s.t.

$2.6b + 1.5c + 2v \le 375$ (calories)
$.2b + .07c + .1v \le 10$ (fat)
$b + .8c \le 50$ (cholestrol)
$b + .8c + 4v \le 400$ (sodium)
$b + c + v \ge 150$ (weight)
$b, c, v \ge 0$

Canonical Form

min $.5b + .07c + v$

s.t.

$2.6b + 1.5c + 2v \le 375$
$.2b + .07c + .1v \le 10$
$b + .8c \le 50$
$b + .8c + 4v \le 400$
$-b - c - v \le -150$
$b, c, v \ge 0$

4. (5 points) Thoroughly explain how you would solve the following standard form linear program. Do **NOT** solve the linear program.

$$\begin{aligned} \max \quad & 4x + 5y + 3z - 2s - t \\ \text{subject to} \quad & \\ & 3x + 2y - z - s = 5 \\ & x + 7z + 5s + t = -23 \\ & -y + 3z + 6s - t = -4 \\ & -16x + 3y - 4z + 6s + 7t = -14 \\ & x, y, z, s, t \geq 0. \end{aligned}$$

Since there is not a natural basis we add artificial variables a_1, a_2, a_3, a_4 so that we have as the constraints

$$\begin{aligned} 3x + 2y - z - s + a_1 &= 5 \\ -x \mp 7z \mp 5s \mp t + a_2 &= +23 \\ +y \mp 3z \mp 6s + t + a_3 &= +4 \\ +16x \mp 3y + 4z \mp 6s \mp 7t + a_4 &= +14 \\ x, y, z, s, t, a_1, a_2, a_3, a_4 &\geq 0 \end{aligned}$$

Subject to these constraints we solve the LP maximizing $-a_1 - a_2 - a_3 - a_4$ via the simplex algorithm. We use a_1, a_2, a_3, a_4 as the ~~original~~ initial basis and $-18 \;\; 0 \;\; 7 \;\; 18 \;\; 7 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 46$ as the initial objective function row. After reaching optimality, if the objective function value is not zero then the original LP is infeasible. If it is feasible, take the portion of the tableau corresponding to the original variables (non-artificial) with the first row $-4 \; -5 \; -3 \;\; 2 \;\; 1 \;\; ?$ and ~~correct~~ make sure the entries over the basic variables are 0, then run the simplex algorithm on this smaller tableau.