Gödel's 2nd Incompleteness Theorem

We outline how Gödel's second incompleteness theorem is formulated and proved. Essentially this theorem states that any "strong enough" axiomatizable theory cannot prove that it is consistent. (Unless it is inconsistent, in which case everything is provable).

Notation: Let $A$ be a recursive set of axioms.

(a) $\text{Ded}^A_{\lambda}(w,x)$ is a numeralwise determined formula of $Q$ and represents the recursive relation:

"$x$ is the Gödel number of a formula of $A$ and $w$ codes a deduction from $A$ of the formula coded by $x$".
(b) \( \text{Thm}_A(x) \) is the formula \( \exists w (\text{Deduct}_A(w,x)) \).

So \( \text{Thm}_A(x) \) asserts "\( x \) codes a consequence of \( A \)."

(c) \( \text{Consis}(A) \) is the sentence

\[ \neg \exists w (\text{Deduct}_A(w, S^{\#0=1})) \]

So \( \text{Consis}(A) \) asserts "\( A \not\vdash 0=1 \)." This implies that \( A \) is consistent. If \( CnA \models \emptyset \) then \( \text{Consis}(A) \) is true iff \( A \) is consistent.

**Definition:** (Peano arithmetic).

PA is a set of axioms in the language \( 0, S, +, \cdot, < \).

PA contains the following sentences:

(a) The sentences in \( Q \),

(b) For each formula \( \varphi \), the induction axiom,

\[ \varphi^0 \land \forall v_1 (\varphi \rightarrow \varphi^v_{Sv_1}) \rightarrow \forall v, \varphi \]

[Fact: \( CnPA \) is not finitely axiomatized.]
Actually, the choice of Peano arithmetic (PA) is not too crucial. The important properties of PA are:

1. PA is consistent
2. \( \mathbb{N} \subseteq \text{PA} \)
3. PA satisfies propositions B-1 and B-2 below.

Thus the results below also hold for theories such as ZF (Set Theory). However, with Set Theory, some technical difficulties arise in interpreting formulas in the language of \( \mathbb{Q} \) in the language of Set Theory - see Enderton for more details.

**Proposition B-1:** Let \( \varphi \), \( \psi \) be arbitrary formulas.

Then

\[
\text{PA} + \text{Thm}_{\text{PA}}(S^\# \varphi) \land \text{Thm}_{\text{PA}}(S^\# \psi) \rightarrow \\
\text{Thm}_{\text{PA}}(S^\#(\varphi \rightarrow \psi)).
\]

In words, PA proves that modus ponens allows a deduction of \( \varphi \) and a deduction of \( \varphi \rightarrow \psi \) to give a deduction \( \psi \).
Proof: A detailed proof of Proposition B-1 would be too long for us to carry out in full - so we shall just say why it's reasonable.

Note that if

\[ \text{Deduct}_{PA} (v_0, s \# \varphi 0) \]

and \( \text{Deduct}_{PA} (w_0, s \# (\varphi \rightarrow \psi) 0) \)

hold (with \( v_0, w_0 \in N \)), then

\[ \text{Deduct}_{PA} (v_0 * w_0 *<\# \psi >, s \# \psi 0) \]

is also true.

So it seems reasonable that

\[ PA + \text{Deduct}_{PA} (v, s \# \varphi 0) \land \text{Deduct}_{PA} (w, s \# (\varphi \rightarrow \psi) 0) \]

\[ \rightarrow \exists x \text{Deduct}_{PA} (x, s \# \psi 0). \]

since \( x = v * w *<\# \psi > \) will work.
Proposition B-2: Let \( \phi \) be an arbitrary formula.

Then

\[
\text{PA} + \text{Thm}_{\text{PA}}(S^{\#\phi}) \rightarrow \text{Thm}_{\text{PA}}(S^{\#\text{Thm}_{\text{PA}}(S^{\#\phi})})
\]

Proof: Omitted. Let's show instead that

\[
\mathcal{N} \models \text{Thm}_{\text{PA}}(S^{\#\phi}) \rightarrow \text{Thm}_{\text{PA}}(S^{\#\text{Thm}_{\text{PA}}(S^{\#\phi})})
\]

So suppose \( \text{Thm}_{\text{PA}}(S^{\#\phi}) \) is true in \( \mathcal{N} \).

Hence, \( \exists \omega \in \mathcal{N} \) such that \( \mathcal{N} \models \text{Deduct}_{\text{PA}}(S^{\omega_0}, S^{\#\phi}) \).

Since \( \text{Deduct}_{\text{PA}} \) is numeralwise determined and \( Q \subseteq \text{Thm}_{\text{PA}} \),

\[
Q \models \text{Deduct}_{\text{PA}}(S^{\omega_0}, S^{\#\phi})
\]

\[
\therefore \quad Q \models \text{Thm}_{\text{PA}}(S^{\#\phi}) \text{ by existential rule}
\]

\[
\therefore \quad \text{PA} \models \text{Thm}_{\text{PA}}(S^{\#\phi}) \text{ since } Q \subseteq \text{PA}
\]

In other words, \( \mathcal{N} \models \text{Thm}_{\text{PA}}(S^{\#\text{Thm}_{\text{PA}}(S^{\#\phi})}) \).
Corollary B-3:
(a) If $\text{PA} + \varphi$ then $Q + \text{Thm}_{\text{PA}} (S^\varphi 0)$
(b) If $\text{PA} + \varphi$ then $\text{PA} + \text{Thm}_{\text{PA}} (S^\varphi 0)$.

Proof: This shown by the argument in support of B-2 on the previous page. (Note the lines marked with :...).

QED Corollary B-3

Definition: Let $\sigma$ be a sentence such that

$$Q + \sigma \leftrightarrow \neg \text{Thm}_{\text{PA}} (S^\sigma 0).$$

$\sigma$ exists by Gödel's Fixed Point Lemma applied to

$$\beta (v_i) = \neg \text{Thm}_{\text{PA}} (v_i).$$

Note $\sigma$ says "I am not a consequence of PA"

or "I am not a PA-theorem".
Proposition B-4: \( \sigma \) is true.

That is to say, \( \mathbb{N} \vdash \sigma \).

Proof: Suppose \( \sigma \) is false. i.e., \( \mathbb{N} \vdash \neg \text{Thm}_{PA}(S^{\#}0) \)

In other words, \( PA \vdash \sigma \).

So \( PA \vdash \neg \text{Thm}_{PA}(S^{\#}0) \) (by defn of \( \sigma \)).

On the other hand,

\( PA \vdash \text{Thm}_{PA}(S^{\#}0) \) by Corollary B-3(6).

And this contradicts the consistency of \( PA \).

Q.E.D. Proposition B-4.

Corollary B-5: \( PA \not \vdash \sigma \)

Pf: Since \( \sigma \) is true. \( \therefore \)

Proposition B-6: \( PA \vdash (\neg \sigma \rightarrow \text{Thm}_{PA}(S^{\#}0)) \)

Pf: By the definition of \( \sigma \) and since \( \mathcal{Q} \subseteq \text{Cn}_{PA} \).
Proposition B-7: \( PA + (-\sigma \rightarrow \text{Thm}_P (S^{\# (-\sigma)}_0)) \)

Proof:

\[ PA + -\sigma \rightarrow \text{Thm}_P (S^{\# -\sigma}_0) \text{ by Proposition B6.} \]

\[ PA + \text{Thm}_P (S^{\# -\sigma}_0) \rightarrow \text{Thm}_P (S^{\# \text{Thm}_P (S^{\# -\sigma}_0)}_0) \]

by Proposition B-2.

\[ PA + \text{Thm}_P (S^{\# (\text{Thm}_P (S^{\# -\sigma}_0) \rightarrow -\sigma)}_0) \text{ by defn of } \sigma \]

and Corollary B-3 (b).

\[ PA + \text{Thm}_P (S^{\# \text{Thm}_P (S^{\# -\sigma}_0)}_0) \rightarrow \text{Thm}_P (S^{\# -\sigma}_0) \]

by Proposition B-1 and modus ponens.

\[ PA + -\sigma \rightarrow \text{Thm}_P (S^{\# -\sigma}_0) \text{ by modus ponens applied twice.} \]

QED Proposition B-7.

Proposition B-8: \( PA + -\sigma \rightarrow -\text{Consis} (PA) \).

Proof: First note \( PA + \text{Thm}_P (S^{\# (\varphi \rightarrow -\varphi \rightarrow 0 = 1)}_0) \).

Now use Propositions B-7, B-8 and two uses of Proposition B-1 to get \( PA + -\sigma \rightarrow \text{Thm}_P (S^{\# 0 = 1}_0) \).

QED.
Corollary B-9: $PA \vdash \text{Consis}(PA)$

Proof: By Corollary B-5 and Proposition B-8.

Q.E.D.

Meta-Theorem: Let $Z$ be any axiomatized theory extending $PA$ (with the set of integers, $0, S, +, \cdot, <$ all definable in $Z$). Then $Z \vdash \text{Consis}(Z)$. 