1. [20 points] Consider the following OpenGL commands:

```
~~glMatrixMode(GL_MODELVIEW);~~
~~glLoadIdentity();~~
~~glRotatef( 90.0, 0.0, 1.0, 0.0 );~~
~~glTranslatef( 2.0, 0.0, 0.0 );~~
~~glScalef( 2.0, 1.0, 1.0 );~~
```

~~What will the 4×4 modelview matrix be equal to after these commands have executed?~~

Consider the transformation

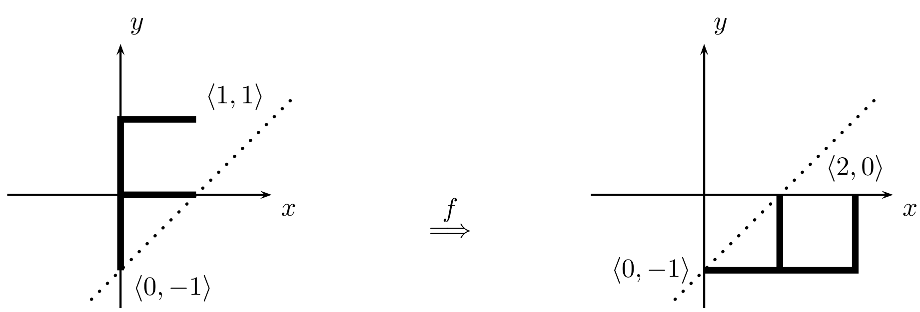
$$R_{\frac{\pi}{2}, \langle 0,1,0 \rangle} \circ T_{\langle 2,0,0 \rangle} \circ S_{\langle 2,1,1 \rangle}$$

Give the 4x4 matrix which represents this transformation over homogeneous coordinates.

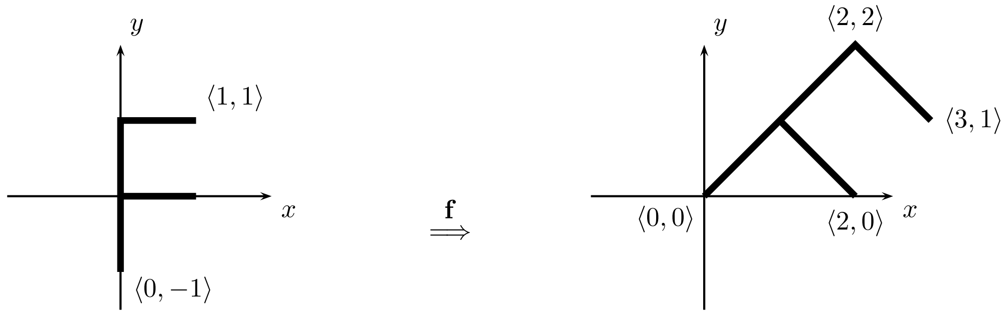
3. [10 points] Give the definition of **linear transformation**.

4. [20 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined to be the transformation that reflects points across the line $y = x - 1$. In particular, it maps a "F" as shown in the picture below.

Give the matrix that represents f over homogeneous coordinates.



1. [20 points] Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined to be the affine transformation that maps an "F" as shown in the picture below.



a. Is f a rigid transformation? Explain why or why not.

b. Express f in the form $f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$ with M a 2×2 matrix.

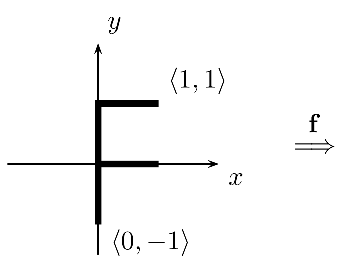
c. ~~Give a sequence of "pseudo" OpenGL commands that will draw the "F" in the position shown on the right. Use commands such as `drawF()` (draws "F" in the position shown on the left), `pglRotatef(···)`, `glTranslate2f(···)`, `glLoadIdentity()`, and `pglScale2f(···)`~~.

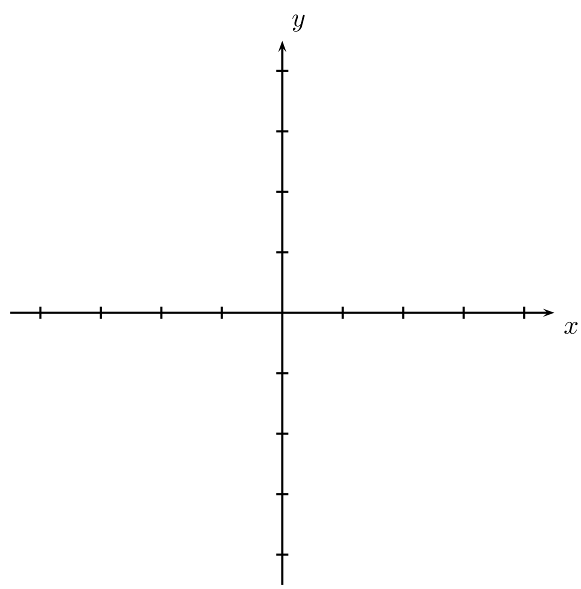
Express the transformation f as a composition of transformations of the forms R_θ, $T_{\vec{u}}$, and $S_{\langle a,b\rangle}$.

2. [15 points] Consider the following 3×3 matrix M that operates on the homogeneous coordinates of points in $\mathbb{R}^2$.

$$\begin{pmatrix} -2 & -2 & 2 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In the empty graph on the right, draw the image of the "F" under the affine map on $\mathbb{R}^2$ that is defined by the matrix M. Draw to scale, and label points as needed.

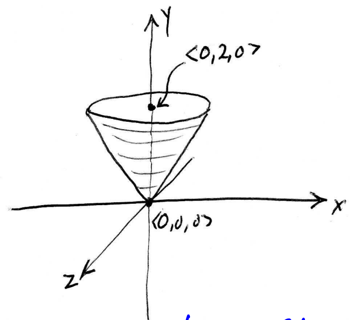




1. [20 points] This problem concerns transformations in $\mathbb{R}^3$. Suppose you are given a function `DrawCone()` that draws a cone of height 1, and base radius 1. This cone drawn by `DrawCone()` is situated centered around the y-axis with its base on the xz plane and the tip of the cone at $\langle 0, 1, 0\rangle$.

a. Give a ~~code fragment~~ description of a transformation that will draw the cone as shown in the figure: the cone is to be drawn upside down, and with height 2 and base radius 2. Its tip is now at the origin; it is still centered around the y-axis.

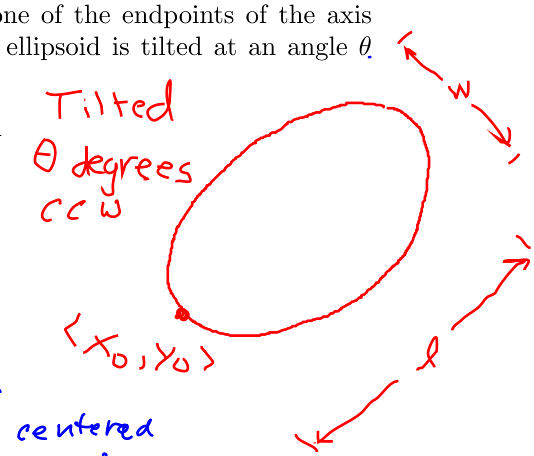
~~Your code fragment that draws the cone~~ may ~~use any of the following pseudo OpenGL commands: `glMatrixMode()`, `glLoadIdentity()`, `glRotatef()`, `glTranslatef()`, `glLoadMatrix()`, `glMultMatrix()`, `glScalef()`, and `DrawCone()`.~~



Your description of the matrix may be given as a composition of rotations ($R_{\theta,\vec{u}}$), translations ($T_{\vec{u}}$) and scalings ($S_{\langle a,b,c\rangle}$).

b. Give a 4×4 homogeneous matrix that gives the same transformation as is used in your answer for part a.

1. [20 points] This problem concerns transformations in $\mathbb{R}^2$. Suppose you are given a function `DrawCircle()` that draws a unit circle centered at the origin (radius equals one). Give a code fragment that will draw an ellipse as shown in the figure. The length of the ellipse is ℓ and the width is w. One endpoint of the ellipse is at $\langle x_0, y_0 \rangle$ in $\mathbb{R}^2$, namely, one of the endpoints of the axis along which the length ℓ is measured. The ellipsoid is tilted at an angle θ. ~~(measured in degrees)~~.



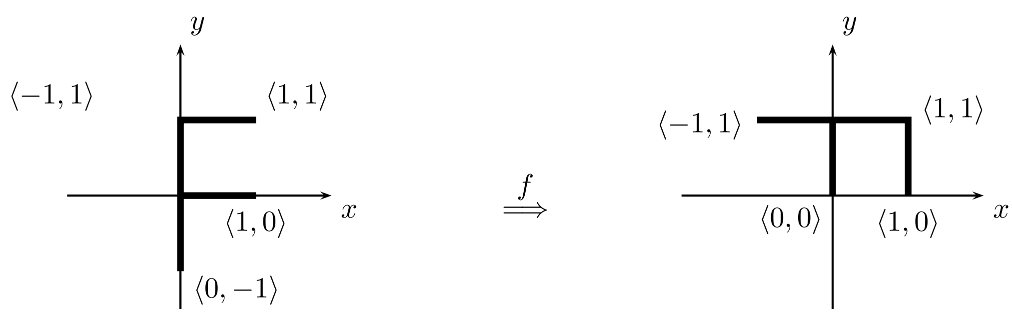
~~Your code fragment that draws the ellipse~~ may ~~use any of the following pseudo OpenGL commands: `glMatrixMode()`, `glLoadIdentity()`, `pglRotatef`~~`()`, ~~`pglTranslatef()`, `pglLoadMatrix()`~~, ~~`pglMultMatrix()`, `pglScalef()`~~, and ~~`DrawCircle()`~~.

Describe the 3x3 matrix which will transform the unit circle centered at the origin to be ellipse as pictured. Describe the matrix as a composition of rotations R_θ, translations $T_{\vec{u}}$ and scalings $S_{\langle a,b \rangle}$.

3. [20 points] A light source is placed at the origin in $\mathbb{R}^3$, and it casts shadows onto the plane defined by $z = -10$. Thus, the plane is like an infinite wall parallel to the xy-plane, placed at $z = -10$.

For $\mathbf{x} = \langle x_1, y_1, z_1 \rangle$ a point in $\mathbb{R}^3$ where $z_1 < 0$, let $A(\mathbf{x}) = \langle x_2, y_2, z_2 \rangle$ be the point on the wall where the shadow of $\mathbf{x}$ is. This means that $z_2 = -10$. Give a 4×4 matrix that represents the transformation A over homogenous coordinates, **or**, prove that there is no such matrix.

1. [36 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps an "F" as shown in the picture below.



a. Is f a linear transformation?

b. Is f a rigid transformation?

c. Is f orientation preserving?

d. Express f in the form $f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$ with M a 2×2 matrix.

e. Now consider the inverse f^{-1} of the transformation f. Give a 3×3 matrix N that represents f^{-1} in homogeneous coordinates.

f. Express f as a generalized rotation $f = R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ by giving the rotation angle θ and the center point $\mathbf{u}$ of the generalized rotation, or explain why this is not possible

3. [12 points] Suppose the function `drawTwoPoints()` draws a point at $\langle 0,0,0\rangle$ and another point at $\langle 1,1,0\rangle$.

a. Consider the ~~sequence of OpenGL commands:~~ transformation A defin

```
glLoadIdentity();
glTranslatef(-1, 0, 0);
glScalef(2, 1, 2);
glRotatef(90, 0, 1, 0);
drawTwoPoints();
```

$$T_{\langle -1,0,0\rangle} \circ S_{\langle 2,1,2\rangle} \circ R_{\frac{\pi}{2},\langle 0,1,0\rangle}$$

~~When the drawTwoPoints() is called, where does the point it draws at $\langle 0,0,0\rangle$ actually get placed (as transformed by the ModelView matrix)? And, where does the point it draws at $\langle 1,1,0\rangle$ get placed?~~

When transformed by A, where do these two points lie?

b. Now consider the slightly different ~~sequence of OpenGL commands:~~

transformation B:

```
glLoadIdentity();
glRotatef(90, 0, 1, 0);
glScalef(2, 1, 2);
glTranslatef(-1, 0, 0);
drawTwoPoints();
```

$$R_{\frac{\pi}{2},\langle 0,1,0\rangle} \circ S_{\langle 2,1,2\rangle} \circ T_{\langle -1,0,0\rangle}$$

~~When the drawTwoPoints() is called, where does the point it draws at $\langle 0,0,0\rangle$ actually now get placed? (You only need to answer about this one point.)~~

Where are the two points placed by B?

4. [12 points] Give short answers to the following questions about the Painter's algorithm:

a. What is the purpose the Painter's algorithm? What problem does it help solve?

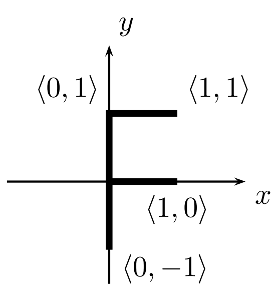
b. Give a short description of how the Painter's algorithm works.

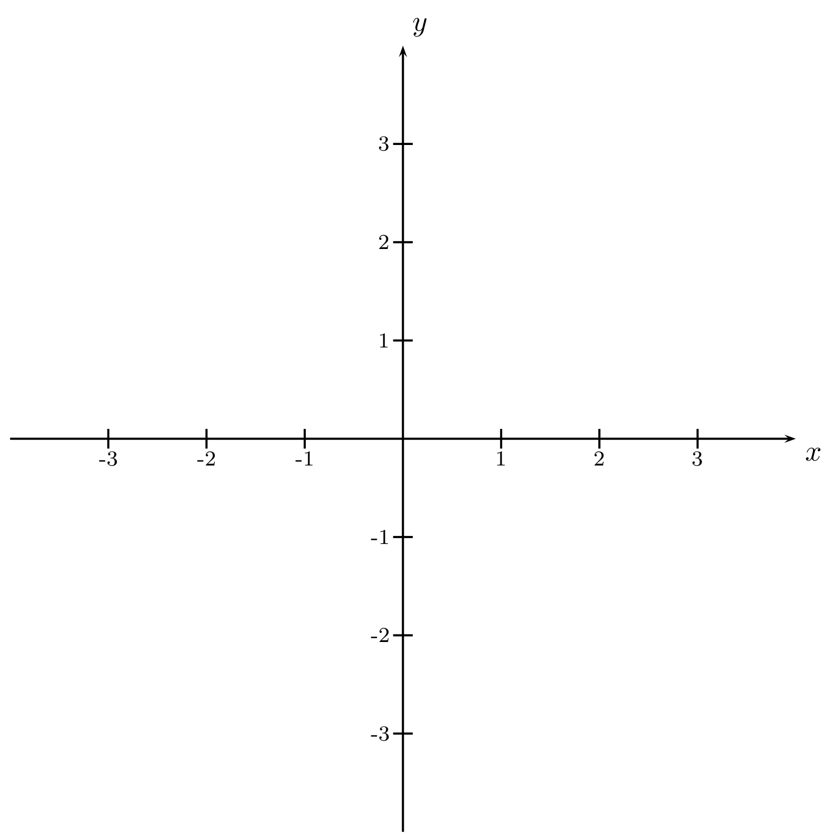
c. Give an example of how the Painter's algorithm can fail to accomplish its purpose.

1. [10 points] Recall that a generalized rotation $R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ is the rigid orientation-preserving transformation which rotates counterclockwise around the point $\mathbf{u}$ (holding the point $\mathbf{u}$ fixed. Express $R_\theta^{\mathbf{u}}$ as a composition of rotations R_φ and translations $T_{\mathbf{v}}$.

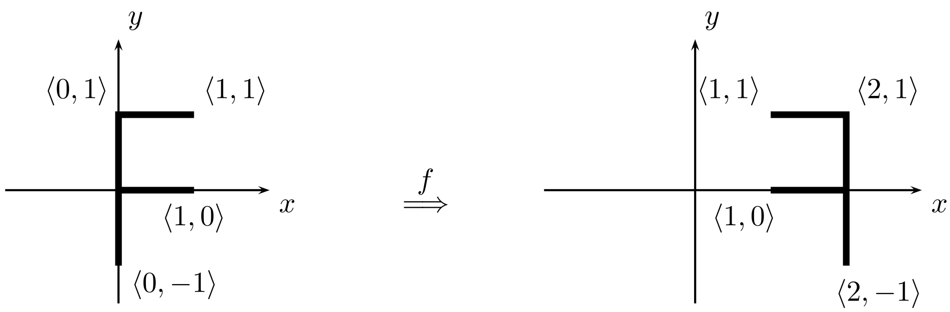
2. [10 points] Consider the following 3×3 matrix M representing a transformation in $\mathbb{R}^2$ over homogeneous coordinates. (Watch out for the lower right entry!) Draw the image of the "F" under this transformation on the large axes to the right. Be sure to label enough points to make your answer clear.

$$M = \begin{pmatrix} 0 & 2 & -2 \\ -4 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$





3. [40 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps an "F" as shown in the picture below.



a. Give a 3×3 matrix which represents f over homogeneous coordinates.

b. Now consider the inverse f^{-1} of the transformation f. Give a 3×3 matrix N that represents f^{-1} in homogeneous coordinates.

c. Express f as a composition of rotations R_θ, translations $T_{\mathbf{u}}$, and/or scalings $S_{\langle a,b\rangle}$, or explain why this is not possible.

d. Express f as a generalized rotation $f = R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ by giving the rotation angle θ and the center point $\mathbf{u}$ of the generalized rotation, or explain why this is not possible

4. [10 points] Suppose we are modelling a Solar System and need to define a transformation A which will place the Earth in the right position. The sun is at the origin. The Earth is distance d from the Sun, lying in the xz plane. We wish to revolve the Earth by angle θ around the Sun (for the time of year). We wish to rotate the Earth on its axis by angle φ (for time of day). We wish to draw the Earth with radius r. (There is no tilt!)

Suppose we have a routine that draws the Earth as a radius 1 sphere centered at the origin. What transformation A needs to be used to place the Earth as desired? Express your answer A as a composition of rotations $R_{\psi,\vec{u}}$. translations $T_{\mathbf{u}}$, and scalings $S_{a,b,c}$.

5. [20 points] A light source is placed at $\langle -10, 0, 0\rangle$ and it casts shadows onto the yz-plane P defined by $x = 0$. The yz-plane is like an infinite wall.

When $\langle x, y, z\rangle$ is a point in $\mathbb{R}^3$ with $-10<x\leqslant 0$, define $A(\langle x, y, z\rangle)$ to be the position of the shadow of the point on the yz-plane. For example, $A(\langle -5, 1, 2\rangle) = \langle 0, 2, 4\rangle$, and $A(\langle -8, 1, 2\rangle) = \langle 0, 5, 10\rangle$

a. Working in ordinary coordinates (not homogeneous) give the formula expressing the mapping $A(\langle x, y, z\rangle) = \langle x', y', z'\rangle$. That is, give formulas for x', y', z' in terms of x, y, z.

b. Give a 4×4-matrix that represents the transformation A over homogeneous coordinates.