1. Let \( \mathbf{x}_1 = \langle -2, 0 \rangle \) and \( \mathbf{x}_2 = \langle 4, 1 \rangle \). Let \( \alpha \) control the linear interpolation (and linear extrapolation) from \( \mathbf{x}_1 \) to \( \mathbf{x}_2 \) by \( \text{Lerp}(\mathbf{x}_1, \mathbf{x}_2, \alpha) \).

   (a) What points are obtained with \( \alpha \) equal to \(-2, -1, 0, \frac{1}{10}, \frac{1}{3}, \frac{1}{2}, 1, \frac{11}{2} \) and \( 2 \)? What value of \( \alpha \) gives the point \( \langle 2, \frac{2}{3} \rangle \)? The point \( \langle 16, 3 \rangle \)? Graph your answers.

   (b) What point \( u \) on the line containing \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) is the closest to the origin? Find the value \( \alpha \) such that \( u = \text{Lerp}(\mathbf{x}_1, \mathbf{x}_2, \alpha) \).

   (c) Suppose the values for \( f(\mathbf{x}_1) = -3 \) and \( f(\mathbf{x}_2) = 3 \) have been set, and we wish to set other values for \( f(z) \) by linear interpolation/extrapolation. What will this set \( f(\langle 2, \frac{2}{3} \rangle) \) equal to?

2. Let \( \mathbf{u} = \text{Lerp}(\mathbf{x}, \mathbf{y}, \alpha) \).

   (a) For what values of \( \alpha \) must \( \mathbf{u} \) be a linear combination of \( \mathbf{x} \) and \( \mathbf{y} \)?

   (b) For what values of \( \alpha \) must \( \mathbf{u} \) be an affine combination of \( \mathbf{x} \) and \( \mathbf{y} \)?

   (c) For what values of \( \alpha \) must \( \mathbf{u} \) be a weighted average of \( \mathbf{x} \) and \( \mathbf{y} \)?

3. Let \( \mathbf{x} = \langle 0, 1 \rangle \), \( \mathbf{y} = \langle 2, 3 \rangle \), and \( \mathbf{z} = \langle 3, 0 \rangle \) in \( \mathbb{R}^2 \). Determine the points represented by the following sets of barycentric coordinates.

   a. \( \alpha = 0, \beta = 1, \gamma = 0 \).
   b. \( \alpha = \frac{2}{3}, \beta = \frac{1}{3}, \gamma = 0 \).
   c. \( \alpha = \frac{1}{3}, \beta = \frac{1}{3}, \gamma = \frac{1}{3} \).
   d. \( \alpha = \frac{4}{5}, \beta = \frac{1}{10}, \gamma = \frac{1}{10} \).
   e. \( \alpha = \frac{4}{3}, \beta = \frac{2}{3}, \gamma = -1 \).

   Graph your answers along with the triangle formed by \( \mathbf{x} \), \( \mathbf{y} \), and \( \mathbf{z} \).

4. Let, again, \( \mathbf{x} = \langle 0, 1 \rangle \), \( \mathbf{y} = \langle 2, 3 \rangle \), and \( \mathbf{z} = \langle 3, 0 \rangle \). Determine the barycentric coordinates of the following points \( \mathbf{u}_1 - \mathbf{u}_4 \):

   a. \( \mathbf{u}_1 = \langle 2, 3 \rangle \).
   b. \( \mathbf{u}_2 = \langle 1\frac{1}{2}, 2\frac{1}{3} \rangle \).
   c. \( \mathbf{u}_3 = \langle \frac{3}{2}, \frac{3}{2} \rangle \).
   d. \( \mathbf{u}_4 = \langle 1, 0 \rangle \).

   The figure also show a point labelled \( \mathbf{u}_5 \). For the barycentric coordinates for \( \mathbf{u}_5 \): Which of \( \alpha, \beta, \gamma \) are positive? Which ones are negative? Which ones are zero?
5. Let \( \mathbf{x} = (0, 0) \), \( \mathbf{y} = (4, 0) \), \( \mathbf{z} = (5, 3) \), and \( \mathbf{w} = (0, 2) \), as shown in the figure. For each of the following values of \( \alpha \) and \( \beta \), what point is obtained by bilinear interpolation? (Give the coordinates of the points a.-d.) Then draw a copy of the quadrilateral, and show the approximate locations of your four answers. (The value \( \alpha \) gives the left-to-right direction; \( \beta \) the bottom-to-top direction.)

a. \( \alpha = 0 \) and \( \beta = 1 \).

b. \( \alpha = \frac{2}{3} \) and \( \beta = 1 \).

c. \( \alpha = \frac{1}{2} \) and \( \beta = \frac{3}{4} \).

d. \( \alpha = \frac{1}{3} \) and \( \beta = \frac{2}{3} \).

6. Suppose a surface patch \( \mathbf{u}(\alpha, \beta) \) in \( \mathbb{R}^3 \) is defined by bilinearly interpolating from four vertices. Derive the following formulas for the partial derivatives of \( \mathbf{u} \):

\[
\frac{\partial \mathbf{u}}{\partial \alpha} = (1 - \beta)(\mathbf{y} - \mathbf{x}) + \beta(\mathbf{z} - \mathbf{w}) = \text{Lerp}(\mathbf{y} - \mathbf{x}, \mathbf{z} - \mathbf{w}, \beta)
\]

\[
\frac{\partial \mathbf{u}}{\partial \beta} = (1 - \alpha)(\mathbf{w} - \mathbf{x}) + \alpha(\mathbf{z} - \mathbf{y}) = \text{Lerp}(\mathbf{w} - \mathbf{x}, \mathbf{z} - \mathbf{y}, \alpha).
\]

In addition, give the general formula for a normal vector to the patch at a point \( \mathbf{u} = \mathbf{u}(\alpha, \beta) \). (It does not need to be a unit vector.)