Math 15A - Discrete Mathematics - Spring 1999
Midterm Exam — May 6 — Answer Key

You may use a single sheet of notes (double-sided, 8½ × 11 paper). You may not use the textbook or a calculator. There are 10 problems worth a total of 360 points. You have one hour and 15 minutes.

Before you begin, write your name & ID number on the cover page, and check that you have all ten problems in your exam.

Good luck!

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1. (30 points) For each formula in the left column, find a logically equivalent formula in the right column.

   \begin{align*}
   \text{c} & \quad (p \rightarrow q) \lor (q \rightarrow p) \\
   \text{e} & \quad (p \rightarrow q) \land (q \rightarrow p) \\
   \text{c} & \quad p \rightarrow (q \rightarrow p) \\
   \text{f} & \quad \sim (q \rightarrow \sim p) \\
   \text{g} & \quad \sim (\sim p \rightarrow q)
   \end{align*}

   \begin{align*}
   \text{a} & \quad p \\
   \text{b} & \quad \sim p \\
   \text{c} & \quad p \rightarrow p \\
   \text{d} & \quad p \land \sim p \\
   \text{e} & \quad p \leftrightarrow q \\
   \text{f} & \quad p \land q \\
   \text{g} & \quad \text{None of the above}
   \end{align*}

2. (24 points) Convert the propositional formula

   \[ p \rightarrow (q \rightarrow (r \leftrightarrow s)) \]

   into an equivalent formula which uses only the connectives \( \land, \lor \) and \( \sim \).

   ANSWER: \( \sim p \lor \sim q \lor (r \land s) \lor (\sim r \land \sim s) \).

   OTHER ANSWERS ARE POSSIBLE.

3. (36 points) Draw a Boolean circuit, using only AND, OR and NOT gates, which has three inputs \( x, y, \) and \( z \) and which outputs 1 (True) if and only if exactly one of its three inputs has value 1.

   ANSWER: I’ll give you the formula: from this it is easy to draw the circuit (by hand, if not by computer).

   \[ \text{OR(AND(AND(x, NOT(y)), NOT(z)), AND(NOT(x), y, NOT(z))), AND(NOT(x), NOT(y), z))} \]
4. (42 points) Let $T(x, y)$ be the predicate “$x$ trusts $y$”. Let $Y(x)$ be the predicate “$x$ is young”, and $A(x)$ be the predicate “$x$ is aged”. Let quantified variables in $(\forall x)$ and $(\exists y)$ range over all people. Express the following formally, using quantifiers, Boolean connectives and the three predicates $T(\cdot, \cdot)$, $Y(\cdot)$, $A(\cdot)$ (but do not use symbols for the set of young or aged people):

(a) Everyone trusts someone.
   ANS: $\forall x \exists y T(x, y)$

(b) No one trusts everyone.
   ANS: $\forall x \exists y \sim T(x, y)$

(c) There is person who is trusted by no one.
   ANS: $\exists x \forall y \sim T(y, x)$

(d) There is person who is trusted by no young people.
   ANS: $\exists x \forall y (Y(y) \to \sim T(y, x))$

(e) There is a person who is trusted by everyone, but who trusts no one.
   ANS: $\exists x ((\forall y T(y, x)) \land (\forall z \sim T(x, z)))$

(f) Some aged people trust no one.
   ANS: $\exists x (A(x) \land \forall y \sim T(x, y))$

(g) Some young people trust all aged people.
   ANS: $\exists x (Y(x) \land \forall y (A(y) \to T(x, y)))$

5. (36 points) Express the following assertions (semi-)formally. You may use quantifiers such as “$\forall$ rational $x$”, “$\exists$ positive integer $y$”, Boolean connections, and predicates such as “$x < y$”, “$x \leq y$”, “$x = y$”, “$x \neq y$”, $x \in \mathbb{Z}$, and the functions addition, subtraction, multiplication, division and square roots. Hints: These assertions should be expanded out into the appropriate definitions. Do not forget to include any necessary conditions such as “$x$ is an integer”.

(a) $x$ divides $y$.
   ANS: $\exists$ integer $q$ s.t. $(x \cdot q = y)$ (You may add $x = 0$ if you wish.)

(b) $r$ is equal to $(x \mod d)$.
   ANS: $0 \leq r < d$ and $\exists$ integer $q$ s.t. $x = qd + r$.

(c) $n$ equals $\lfloor x \rfloor$.
   ANS: $n \leq x < n + 1$.

(d) $x$ is a prime.
   ANS: $x > 1$ and $\forall$ positive integers $s, r$ (if $s \cdot r = x$ then $s = 1$ or $r = 1$).
6. (36 points) (a) Let \( n \) be the integer with binary (base 2) representation 110101. Express \( n \) in base 10:

\[ \text{ANSWER: } 53 \]

(b) Let \( m \) be the negative integer which has 8-bit, two’s-complement, binary representation 11110110. Express \( m \) in base 10.

\[ \text{ANSWER: } -10 \]

7. (36 points) Let the sequence \( a_0, a_1, a_2, \ldots \) be defined by the values \( a_0 = 1 \) and \( a_{n+1} = \sqrt{2} \cdot a_n \) for all \( n \geq 0 \).

To answer the next questions, you should give exact expressions, including the \( \sqrt{2} \) as necessary. (Do not use a calculator to approximate in decimal form.)

(a) What is the value of \( a_{10} \)? ANSWER: \( (\sqrt{2})^{10} = 2^5 = 32 \).

(b) What is the value of \( \sum_{i=0}^{11} a_i \)? ANSWER: \( \frac{(\sqrt{2})^{12} - 1}{\sqrt{2} - 1} = \frac{63}{\sqrt{2} - 1} = 63(\sqrt{2} + 1) \).

The next three problems ask you to give proofs. Your answers will be graded on form as well as content: be sure to explain all your steps, use correct grammar, and follow the conventions for writing proofs.

8. (40 points) Either prove or disprove the following assertion:

**Assertion:** For all integers \( n \), \( n^3 - n \) is divisible by 3.

**ANSWER:** This true. The most straightforward proof uses a proof by cases based on \( n \mod 3 \).

**Proof:** Let \( n \) be an integer. By basic algebra, it will suffice to prove that \( n^3 \mod 3 \) is equal to \( n \mod 3 \). This is because if \( n = 3k + r \) and \( n^3 = 3m + r \) for some integers \( k \) and \( m \), then \( n^3 - n = 3(m - k) \) and is thus divisible by 3.

**Case 1:** \( n = 3k \) for some integer \( k \). Then \( n^3 = 3(9k^3) \) is also a multiple of 3.

**Case 2:** \( n = 3k + 1 \) for some integer \( k \). Then \( n^3 = 3(3^2k^3 + 3^2k^2 + 3k + 1) + 1 \) by basic algebra. So \( n^3 \mod 3 = n \mod 3 \).

**Case 3:** \( n = 3k + 2 \) for some integer \( k \). Then \( n^3 = 3(3^2k^3 + 2 \cdot 3^2k^2 + 2^2 \cdot 3k + 2) + 2 \) by basic algebra. So \( n^3 \mod 3 = n \mod 3 \).

**Q.E.D.**
9. (40 points) Either prove or disprove the following assertion:

**Assertion:** For all rational numbers \(a\) and \(b\), \(a/b\) is a rational number.

**ANSWER:** This is false. Taking \(b = 0\) and \(a\) any rational number yields a counterexample.

10. (40 points) Prove by induction on \(n\):

**Theorem:** Let \(a_1 = 1\) and \(a_{n+1} = \frac{n^2}{n+1} a_n\) for all integers \(n \geq 1\). Then, for all positive integers \(n\), \(a_n = \frac{(n-1)!}{n}\).

**Proof:** We prove the assertion by induction on \(n \geq 1\).

**Base case:** \(n = 1\). \(e_1 = 1\). And \(\frac{0}{1} = \frac{1}{1} = 1\).

**Induction step:** Suppose \(a_n = \frac{(n-1)!}{n}\) is true. We shall prove that \(a_{n+1} = \frac{n!}{n+1}\). We argue as follows:

\[
a_{n+1} = \frac{n^2}{n+1} \cdot a_n
\]

\[
= \frac{n^2}{n+1} \cdot \frac{(n-1)!}{n} \quad \text{by the ind. hyp.}
\]

\[
= \frac{n!}{n+1} \quad \text{by basic algebra.}
\]

**Q.E.D.**