

Name: _____

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Math 15A - Discrete Mathematics - Spring 1999

Answers for Midterm Review Problems — May 3

On the midterm, you may use a *single* sheet of notes (double-sided, $8\frac{1}{2} \times 11$ paper). You may not use the textbook or a calculator. This sheet may not be shared with other students: you must have your own separate sheet of notes.

The midterm will cover all the course material to-date, through section 4.3.

1. (a) Consider the sequence $1, -2, 4, -8, 16, -32, \dots$. Letting $a_0 = 1$, $a_1 = -2$, etc., give a general formula for a_i .

$$\text{ANSWER: } a_i = (-2)^i.$$

- (b) Give a formula for $\sum_{i=0}^n a_i$.

$$\text{ANSWER: } \frac{(-2)^{n+1} - 1}{-3}.$$

2. (a) Express the integer 85 in binary (base 2) notation.

$$\text{ANSWER: } 1010101.$$

- (b) Express the negative integer -85 in binary notation (using a 8-bit two's complement representation).

$$\text{ANSWER: } 10101011$$

3. Convert the propositional formula

$$p \rightarrow (q \rightarrow (r \rightarrow (s \rightarrow t)))$$

into an equivalent formula using only the connectives \wedge , \vee and \sim .

$$\text{ANSWER: } \sim(p \wedge q \wedge r \wedge s) \vee t.$$

$$\text{ANOTHER ANSWER: } \sim p \vee \sim q \vee \sim r \vee \sim s \vee t$$

4. Express the formula $(p \leftrightarrow q) \leftrightarrow r$ as a Boolean circuit using AND, OR and NOT gates.

HINT: use the fact that $(p \leftrightarrow q)$ is the same as $(p \text{ AND } q) \text{ OR } (\text{NOT } p \text{ AND NOT } q)$. The output of that circuit computing $(p \leftrightarrow q)$ is input along with r into another copy of the same circuit. The resulting circuit has 10 gates.

5. Let $L(x, y)$ be the predicate “ x loves y ”, and let $H(x)$ be the predicate “ x is handsome”. Let quantified variables in $(\forall x)$ and $(\exists y)$ range over all people. Express the following formally using quantifiers, Boolean connectives and the two predicates:

- (a) Someone loves everyone.

$$\text{ANSWER: } \exists x \forall y L(x, y).$$

(b) Everyone is loved by someone.

ANSWER: $\forall x \exists y L(y, x)$.

(c) Some people love no one.

ANSWER: $\exists x \forall y \sim L(x, y)$.

(d) Any handsome person is loved by everyone.

ANSWER: $\forall x (H(x) \rightarrow \forall y (L(y, x)))$.

(e) There is a handsome person who is loved by no one.

ANSWER: $\exists x (H(x) \wedge \forall y (\sim L(y, x)))$.

(f) For a person to be loved by everyone it is necessary that they be handsome.

ANSWER: $\forall x ((\forall y (L(y, x))) \rightarrow H(x))$.

6. Which of the following are tautologies?

(a) $(\sim(p \wedge \sim q)) \leftrightarrow (\sim p \vee q)$

(b) $p \rightarrow ((p \rightarrow q) \rightarrow p)$

(c) $(p \leftrightarrow (q \leftrightarrow r)) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow r)$.

ANSWERS: ALL THREE ARE TAUTOLOGIES.

7. Express the following assertions (semi-)formally. You may use quantifiers such as “ \forall rational x ”, “ \exists positive integer y ”, Boolean connections, and predicates such as “ $x < y$ ”, “ $x \leq y$ ”, “ $x = y$ ”, “ $x \neq y$ ”, and functions addition, subtraction, multiplication, division and square roots.

(a) There is a smallest positive integer.

ANSWER: \exists a positive integer x s.t. \forall positive integers y , $x \leq y$.

(b) There is no smallest positive rational number.

ANSWER: \forall rationals x , \exists a rational y s.t. $y < x$.

(c) For any irrational $x < y$, there is a rational z between x and y .

ANSWER: \forall irrationals $x < y$, \exists a rational z s.t. $x < z < y$.

(d) For an integer n to be prime, it is sufficient that n has at most one divisor which is greater than or equal to \sqrt{n} .

ANSWER: This problem is harder than I intended it to be. To answer it the way I asked, you will need to expand out the definition of “prime” and “divides”. The medium length answer is:

\forall integers n , if $(\forall$ integers x, y , if $\sqrt{n} < x, y$ and $x|n$ and $y|n$ then $x = y$), then n is prime.

8. Which of the assertions (a)-(d) of the previous problem are true?

ALL OF THEM ARE TRUE.

9. Prove or disprove the following assertion:

Assertion: Let n be an integer. Then $n^3 \bmod 8$ is equal to $n \bmod 8$.

THIS IS TRUE. PROVE IT BY CASES! IN THE MOST STRAIGHT-FORWARD PROOF THERE ARE 4 CASES DEPENDING ON THE VALUE OF $n \bmod 8$.

Proof. Let n be an integer. Since n is odd, $n \bmod 8$ is odd and hence equal to 1, 3, 5, or 7. (If $n \bmod 8$ were even, then $n = 8k + p$ for some integers k, p with p even, and then n is the sum of two even integers and thus even.)

Case 1 Assume $n \bmod 8$ equals 1. Then $n = 8p + 1$ for some integer p . And $n^3 = 8(64n^3 + 3 \cdot 8n^2 + 3n) + 1$, so $n^3 \bmod 8$ is also equal to 1.

Case 2 Assume $n \bmod 8$ equals 3. Then $n = 8p + 3$ for some integer p . And $n^3 = 8(64n^3 + 3 \cdot 3 \cdot 8n^2 + 3 \cdot 3^2n + 3) + 3$, so $n^3 \bmod 8$ is also equal to 3.

Case 3 Assume $n \bmod 8$ equals 5. *This is similar to the previous cases and I omit the argument from this answer sheet.*

Case 4 Assume $n \bmod 8$ equals 7. *This is similar to the previous cases and I omit the argument from this answer sheet.*

10. Prove or disprove the following assertion: (You may assume without proof that $\sqrt{2}$ is irrational if you need to. Any other needed irrational number should be proved to be irrational.)

Assertion: For all irrational numbers a, b with $a + b \neq 0$, the number $a/(a + b)$ is irrational.

This assertion is false. To disprove it, it is sufficient to find a counterexample. Choose $a = b = \sqrt{2}$. Then $a/(a + b) = 1/2$ by basic algebra, and this is rational. q.e.d.

11. Prove the following by induction.

Theorem: $\sum_{n=0}^{k-1} 3n(n + 1) = k^3 - k$.

Proof: Let $P(k)$ be the assertion that the theorem holds for k . We prove this for all integers $k > 0$, by induction on k .

- Base case: $k=0$. The sum is empty when $k = 0$ and by convention, the empty sum has value 0. Likewise, $0^3 - 0 = 0$. So $P(0)$ is true.
- Induction step: Assume $k \geq 0$, $k \in \mathbb{Z}$ and that $P(k)$ is true. We must prove $P(k + 1)$ holds.

The summation is equal to

$$\begin{aligned} \sum_{n=0}^k 3n(n + 1) &= \sum_{n=0}^{k-1} 3n(n + 1) + 3k(k + 1) \\ &= k^3 - k + 3k^2 + 3k && \text{by the ind. hyp.} \end{aligned}$$

The righthand side is equal to

$$(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1.$$

These are clearly equal, so $P(k + 1)$ holds.

Q.E.D.