Math 15A - Discrete Mathematics - Spring 1999
Midterm Review Problems — May 3

On the midterm, you may use a single sheet of notes (double-sided, 8 1/2 x 11 paper). You may not use the textbook or a calculator. This sheet may not be shared with other students; you must have your own separate sheet of notes.

The midterm will cover all the course material to-date, through section 4.3.

1. (a) Consider the sequence 1, −2, 4, −8, 16, −32, …. Letting $a_0 = 1$, $a_1 = −2$, etc., give a general formula for $a_i$.
   (b) Give a formula for $\sum_{i=0}^{n} a_i$.

2. (a) Express the integer 85 in binary (base 2) notation.
   (b) Express the negative integer −85 in binary notation (using a 8-bit two’s complement representation).

3. Convert the propositional formula
   \[ p \rightarrow (q \rightarrow (r \rightarrow (s \rightarrow t))) \]
   into an equivalent formula using only the connectives $\land$, $\lor$ and $\neg$.

4. Express the formula $(p \leftrightarrow q) \leftrightarrow r$ as a Boolean circuit using AND, OR and NOT gates.

5. Let $L(x, y)$ be the predicate “$x$ loves $y$”, and let $H(x)$ be the predicate “$x$ is handsome”. Let quantified variables in $(\forall x)$ and $(\exists y)$ range over all people. Express the following formally using quantifiers, Boolean connectives and the two predicates:
   (a) Someone loves everyone.
   (b) Everyone is loved by someone.
   (c) Some people love no one.
   (d) Any handsome person is loved by everyone.
   (e) There is a handsome person who is loved by no one.
   (f) For a person to be loved by everyone it is necessary that they be handsome.

6. Which of the following are tautologies?
   (a) $(\neg(p \land \neg q)) \leftrightarrow (\neg p \lor q)$
   (b) $p \rightarrow ((p \rightarrow q) \rightarrow p)$
   (c) $(p \leftrightarrow (q \leftrightarrow r)) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow r)$. 
7. Express the following assertions (semi-)formally. You may use quantifiers such as “∀ rational \( x \), “∃ positive integer \( y \)”, Boolean connections, and predicates such as “\( x < y \)”, “\( x \leq y \)”, “\( x = y \)” “\( x \neq y \)”, and functions addition, subtraction, multiplication, division and square roots.

(a) There is a smallest positive integer.

(b) There is no smallest positive rational number.

(c) For any irrational \( x < y \), there is a rational \( z \) between \( x \) and \( y \).

(d) For an integer \( n \) to be prime, it is sufficient that \( n \) has at most one divisor which is greater than or equal to \( \sqrt{n} \).

8. Which of the assertions (a)-(d) of the previous problem are true?

9. Prove or disprove the following assertion:

Assertion: Let \( n \) be an odd integer. Then \( n^3 \mod 8 \) is equal to \( n \mod 8 \).

10. Prove or disprove the following assertion: (You may assume without proof that \( \sqrt{2} \) is irrational if you need to. Any other needed irrational number should be proved to be irrational.)

Assertion: For all irrational numbers \( a, b \) with \( a + b \neq 0 \), the number \( a/(a + b) \) is irrational.

11. Prove the following by induction.

Theorem: \( \sum_{n=0}^{k-1} 3n(n + 1) = k^3 - k \).