Math15A - Discrete Mathematics
Suggestions on How to Write Proofs
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One way to define mathematics is that is the study of structures which can be precisely and unambiguously defined. These structures include things like the integers, real numbers, geometrical objects, functions, relations, etc. Since these objects can be unambiguously defined, it is possible to establish truths about these objects which are known to be true *beyond any reasonable doubt*. Mathematical **proofs** are used to establish these kinds of mathematical truths: the purpose of proof is convince both its author and its readers that certain theorems are true.

In Math 15A, we will teach (mainly by example) various approaches to writing proofs: these approaches tend to be oriented towards how to use common proof strategies, and how to obey the generally accepted conventions of writing proofs.

Reading proofs: Before one can write proofs, one needs to be able to *read* proofs. Here the general guidelines are:

- Read the proof *sceptically*. An effective proof should drag you to a conclusion even if you are disinclined to accept the conclusion. Try not to let the proof's author fool you by skipping any steps: if steps *are* skipped, then you have to fill them in yourself.

  If the author uses any unproved assumptions, then you should notice this.

  The bumper sticker "Question Authority" definitely applies to the authors and readers of mathematical proofs: you should expect a proof to convince *you*, and to not need to take assertions on faith or authority.

- Before you start reading the proof, think about what needs to be proved and how one would naturally go about trying to prove it. This will help you understand why certain steps are taken (and may help you fill in the steps skipped by the author).

- Read every word of the proof, and understand every assertion made in the proof. Proofs are generally tersely worded and one cannot understand a proof if a single line is misunderstood.

- You have the right to demand of the proof author that the proof be written correctly, using correct grammar, complete sentence, and correct mathematical syntax.

Some general guidelines for writing proofs are:

- Before you try to write a proof, express the theorem in (semi-)formal language, including all the quantifiers, etc., in proper syntax.

- Start the proof with the word "**Proof**". End the proof with "Q.E.D.", Latin for "it is demonstrated".

- Write your proof in correct grammar: use complete sentences, follow correct English grammar, use conventional and syntactically correct mathematical expressions.

- State your assumptions clearly.
- Explain every step that is taken in the proof. For instance, if you are about to embark on a proof by contradiction, say that the proof is by contradiction and state precisely the assumption that is being made (that will disproved by deriving a contradiction). Likewise, a proof by induction should clearly and precisely state the statement being proved by induction.
- Keep your audience in mind and write the proof at level appropriate for your readers. In a course like Math 15A, the proofs should be written more-or-less in the style used in the text or presented in class. However, both the text and the instructor may get sloppy at times and present less detail than they really should (the instructor can get away with this partly since he can explain things verbally too, and partly since students are sometimes reluctant to ask questions). Your proofs may need to use a little more detail than the text and the instructor.
- Don't use the same variable for two different purposes.
- Avoid logical fallacies!

Finding proofs can be a tremendously hard process and require a lot of both creativity and luck. Even after the creative insight, it may take a lot of work to structure the proof clearly and actually write out the proof.

In Math 15A, you will be asked to prove only relatively straightforward theorems. The following guidelines should be helpful for finding proofs:

- To prove $(\forall x)A(x)$ or $(\forall x \in D)A(x)$, assume that x is an arbitrary object (or that x is an arbitrary object in D in the second case) and then prove $A(x)$ must hold.
- To prove $(\exists x)A(x)$ or to prove $(\exists x \in D)B(x)$, it is sufficient to find some particular x and prove that $x \in D$ (in the second case) and that $A(x)$ holds. Note that it may require some creativity or inspiration to decide what to use for x.
- To prove $A \wedge B$, prove both A and B.
- To prove $A \vee B$, prove one of A or B. Often a disjunction will be proved using a "proof by cases".
- To prove $A \rightarrow B$, you may:
  - Assume A is true and prove B.
  - Assume $\sim B$ is true and prove $\sim A$. (I.e, prove the contrapositive of $A \rightarrow B$.)
  - Assume both A and $\sim B$ are true and derive a contradiction.
- (Proof by contradiction). To prove A, you may assume $\sim A$ and derive a contradiction.
- To prove $A \leftrightarrow B$, you must prove both $A \rightarrow B$ and $B \rightarrow A$.
- (Proof by cases). If you are trying to prove C and you know (or can prove) that $A \vee B$ is true, then it sufficient to prove $A \rightarrow C$ and $B \rightarrow C$. Often one can take B to be $\sim A$.

  Proof by cases can be done with any number of cases, not just two.
- (Induction) To prove $A(x)$ for x any integer, use induction on x. Prove $A(0)$ and then prove $\forall z(A(z) \rightarrow A(z+1))$.