Math 160A - Introduction to Mathematical Logic

Final Examination

March 19, 2002

Write your name or initials on every page before beginning the exam.

You have three hours. There are 14 problems worth a total of 280 points (20 points each). You may not use computers, the internet, notes, the textbook, or other materials during this exam. Please show your work. Good luck!

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1. Translate the following English sentences into first order logic. Use the language with the following symbols: binary predicate L, unary function m and equality ($=$). You may use the usual conventions on abbreviating formulas.

$L(x, y)$ - "x likes y".

$m(x)$ - x's mother.

a. No one likes everyone.

b. Everyone likes someone. (I.e., no one dislikes everyone.)

c. Everyone likes their mother.

d. Everyone is liked by their mother.

e. Everyone is liked by their mother's mother.

f. No one is their own mother.

g. Two people with the same mother like each other.

h. Every mother likes at least one of her children.

i. Some mothers have only one child.

2. For each, indicate whether true or false.

____ **a.** $P \to Q \to R \models\dashv Q \wedge P \to R$.

____ **b.** $P \to Q \to P \models\dashv S \to R \to S$.

____ **c.** $P \wedge (Q \vee R) \models\dashv (P \wedge Q) \vee (R \wedge P)$.

____ **d.** $P \vee (Q \wedge R) \models\dashv (P \vee Q) \wedge (R \vee P)$.

3. Which of the following are first-order tautologies? (Indicate your answer clearly!)

a. $\forall x P(x) \to \exists x P(x) \to \forall x P(x)$

b. $\exists y(\forall x P(x, y) \to \forall x P(x, y))$

c. $\forall x P(x) \to P(x)$

4. For each of the following formulas, give an equivalent formula in disjunctive normal form.

a. $P \leftrightarrow Q$

b. $\neg P \leftrightarrow Q$.

c. $(P \vee \neg Q) \wedge (\neg P \vee Q)$.

5. Is $\{\rightarrow, \leftrightarrow\}$ complete? Give a proof for your answer!

6. List all the 0-ary and unary propositional connectives.

7. Prove **a.** and **b.**

a. $\exists x(P(x) \to Q(x)) \nvDash \exists x P(x) \to \exists x Q(x)$.

b. $\forall x P(x) \to \forall x Q(x) \nvDash \forall x(P(x) \to Q(x))$.

8. Prove $\exists x P(x) \to \exists x Q(x) \vDash \exists x(P(x) \to Q(x))$.

9. Show that the following sets are definable in the structure $(\mathbb{R}, 0, +, \cdot)$.

a. $\{x : x \geqslant 0\}$.

b. $\{x : x = 1\}$.

c. $\{x : x \geqslant 1\}$.

10. The first-order language for directed graphs (loops allowed) has a single binary predicate E and equality $(=)$. $E(x, y)$ means there is an edge from x to y. The term "loop-free" means there is no edge from any vertex to itself.

a. Prove that the set of loop-free graphs is an elementary class (an EC).

b. Prove that the set of infinite, loop-free graphs is an elementary class in the wide sense (an EC_Δ).

11. For each of the following, indicate whether it is a logical axiom from Λ.

a. $v_1 = v_1$.

b. $\forall v_1(v_1 = v_1)$.

c. $\forall v_1(v_1 = v_2 \to v_2 = v_1)$.

d. $\forall v_1 \exists v_2(v_1 \leqslant v_2) \to \exists v_2(v_2 \leqslant v_2)$.

e. $\forall x Q(x) \to Q(x)$.

f. $\forall x(Q(x) \to Q(x))$.

12. Give (explicitly) deductions that prove the following:

a. $\forall x P(x, y) \vdash P(y, y)$.

b. $\vdash Q(y) \to \exists x Q(x)$.

13. For each, indicate whether true or false.

____ **a.** $P(x) \vDash \forall x P(x)$.

____ **b.** $\forall x \forall y P(x,y) \vDash \forall x P(x,x)$.

____ **c.** $\forall x P(x,y) \vDash \exists v \forall u P(u,v)$.

Now, for each *false* one, provide a counterexample that shows it is false.

14. Suppose that Γ and Δ are effectively enumerable sets of propositional formulas. Prove that the set $\{\phi \in \Delta : \Gamma \vDash \phi\}$ is effectively enumerable.